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**Special Tortkara algebras and assosymmetric algebras**

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# NORMATIVE REFERENCES

This dissertation uses references to the following standards:

* SN RK 4.04-04-2013, Street lighting of urban communities and rural settlements.
* Rules for awarding academic degrees dated March 31, 2011 No.127
* GOST 7.32-2001 (changes from 2006). Research report. Structure and rules of registration.
* GOST 7.1-2003. Bibliographic record. Bibliographic description. General requirements and rules for compilation.

# NOTATIONS

**The following notations are used in this dissertation:**

|  |  |  |
| --- | --- | --- |
|  |  | a variety of algebras defined by some set of identities |
|  |  | a class of all algebras of the types defined by |
|  |  | a class of all algebras of the types defined by |
|  |  | a free nonassociative algebra generated by set |
|  |  | a free Zinbiel algebra generated by set |
|  |  | a free Tortkara algebra generated by set |
|  |  | a free special Tortkara algebra generated by set |
|  |  | n-th homogenous part of |
|  |  | the shuffle product of and |
|  |  | is for |
|  |  | is a skew-right-commutative element of the form  for |
|  |  | the -th commutator ideal of |
|  |  | is spanned by |
|  |  | is spanned by |
|  |  | the ideal of generated by . |
|  |  | the space of all cocycles |
|  |  | the space of all coboundaries |
|  |  | the second cohomology space defined as |
|  |  | the annihilator of |
|  |  | the automorphism group of the algebra |
|  |  | *j*-th *i*-dimensional nilpotent non-pure assosymmetric algebra with identity |
|  |  | *j*-th *i*-dimensional nilpotent pure assosymmetric algebra without identity |
|  |  | *i*-dimensional algebra with zero product |
|  |  | *j*-th *i*-dimensional central extension of |

# INTRODUCTION

The presented dissertation is devoted to the special Tortkara algebras and nilpotency of assosymmetric algebras associated with Lie ideals and their algebraic classification in low dimensions.

The class of Tortkara algebras is a new class of nonassociative algebras discovered by A. Dzhumadil’daev [1]. Nonassociative algebras play an important role in many areas of mathematics. It is known that nonassociative algebras, such as Lie and Jordan algebras, arose within the framework of physics and have been extensively developed due to their applications in this field of science. Therefore, the classical theory of nonassociative algebras is mainly based on the study of Lie and Jordan algebras.

Let be a variety of algebras defined by some set of identities. We define and as the classes of all algebras of the types and defined by the anticommutator and the commutator respectively on the same vector space of for all These two products usually connect two different known varieties of algebras, sometimes leading to new interesting classes of algebras. A classic example of such an approach is the variety of associative algebras. Recently, there has been a wide interest in studying other new types of nonassociative algebras over the commutator and anticommutator, such as Novikov, assosymmetric, bicommutative, Leibniz, and other algebras [2 – 6].

The commutator and anticommutator algebras of an associative algebra are known to satisfy Jacobi and Jordan identities, respectively. According to the well-known Poincare-Birkhoff-Witt (PBW) theorem, there are two independent identities, namely anticommutativity and Jacobi identities, which provide a complete list of identities for the commutator algebra of an associative algebra. This means that every identity satisfied by the commutator product in every associative algebra is a consequence of anticommutativity and Jacobi identities. However, the situation is different in the case of the anticommutator, as there is no embedding theorem. P. Cohn in [7] showed that a free special Jordan algebra with three generators has an exceptional homomorphic image. Consequently, there is no analogue of the PBW theorem for Jordan algebras. Also, it is known that the Glennie identity of degree eight exists, which is not a consequence of commutativity and Jordan identities. In this situation, many different interesting questions arise: studying the speciality of Jordan algebras, finding special identities, and others. In 1956 Shirshov proved that every Jordan algebra with two generators is special. This result gave further development of the theory of Jordan algebras. In this dissertation, we prove analogies of Cohn’s and Shirshov’s theorems for the free special Tortkara algebras.

Another important question in this line of research is the search for criteria to determine whether an element of free algebra is a Lie or a Jordan element. Let be a free algebra on a set of An element of the algebra is called a Lie element if it can be expressed by elements of *X* in terms of commutators. Similarly, an element of is called a Jordan element if it can be expressed by elements of in terms of anticommutators. There are two well-known Lie criteria for free associative algebras: the Specht-Wever-Dynkin criterion [8, 9] and the Friedrich criterion [10]. Jordan elements in free associative algebra were described by P. Cohn [7, p. 259] only for the set of generators containing no more than three elements. He showed that an element is Jordan if and only if it is symmetric under the involution map. Using this criterion, some structural results concerning the theory of Jordan algebras were obtained. Based on Cohn’s result D. Robbins developed the study of Jordan elements in the free associative algebras [11]. But here, we give both Lie and Jordan criteria for elements in a free Zinbiel algebra.

A. Dzhumadil’daev proved that Zinbiel algebra under commutator satisfies the anticommutativity and Tortkara identities [1, p. 3911]. M. Bremner in [12], using representation theory, studied special identities in terms of the triple product of Tortkara defined as in a free Zinbiel algebra and discovered identities in degrees five and seven in terms of the triple product. Recently, some geometric interpretations of Tortkara algebras have emerged in data science [13, 14]. The algebraic and geometric classifications of 5- and 6-dimensional Tortkara algebras were obtained in [15 – 17]. P. Kolesnikov in [18] shows that the class of all special Tortkara algebras does not form a variety. In the anticommutator case, he showed that there exists a homomorphic image of a free anticommutator algebra from a single generator, which is not embedded in the anticommutator algebra of the Zinbiel algebra. In addition, he asked a question about the maximum number of free generators for which all homomorphic images of a free special Tortkara algebra are special. The first part of the dissertation is devoted to the study of the above questions for Zinbiel and Tortkara algebras.

The second approach in our investigation is determining the structure of a Lie-admissible algebra when its related Lie algebra satisfies certain properties. Many of the properties of commutator subgroups had analogues in the theory of associative algebras in [19], with a suitable definition of “commutator ideals”. Jennings in [19, p. 341] extended the concepts of a “nilpotent group” and a “solvable group” to a ring. He proved that if is an associative algebra over a field with characteristics not equal to 2, if the associated Lie algebra is solvable, then is solvable. Moreover, he obtained that if is an associative algebra whose associated Lie algebra is nilpotent, then the ideal of is generated by the set is nilpotent. In [20] established that if is an associative algebra over a field whose associated Lie algebra is solvable, and if the characteristic of is neither 2 nor 3, then is nil. If the associated Lie algebra of the associative algebra over a field of characteristic is either nilpotent or solvable with , then the ideal is nil of bounded index [21]. Assosymmetric algebras are introduced by Kleinfeld which come close to being associative [22]. Assosymmetric algebras as associative algebras under commutator are Lie-admissible algebras [23]. Kleinfeld proved that an assosymmetric ring of characteristic different from 2 and 3, without ideals such that is associative. In addition, assosymmetric algebras were studied in [24 – 26]. The basis of free assosymmetric algebras was presented in [27]. Pokrass and Rodabaugh proved that each solvable assosymmetric ring of characteristics different from 2 and 3 is nilpotent [28]. We are continuing the investigation of Lie-admissible algebras such as assosymmetric algebras in terms of their associated Lie algebras.

The third considered problem is the classical problem in nonassociative algebra theory is to classify (up to isomorphism) the algebras of dimension arising from a given variety described by a set of polynomial identities. It is common to concentrate on small dimensions, and there are two basic classification approaches: algebraic and geometric. These two methodologies have been used to study associative, Jordan, Lie, Leibniz, Zinbiel, and others, see [29 – 35] and references therein. We focus on the classification of the finite-dimensional nilpotent assosymmetric algebras. The key step of the method of algebraic classification of nilpotent assosymmetric algebras is the calculation of central extensions of small dimensional algebras. Firstly, Skjelbred and Sund devised a method for classifying nilpotent Lie algebras employing central extensions [36]. Moreover, the method was used to describe different varieties of nilpotent algebras of small dimensions such as the -dimensional nilpotent: associative algebras, Novikov algebras, bicommutative algebras, and Zinbiel algebras [32, p. 4, 37 – 39], all the -dimensional nilpotent Jordan algebras [34, p. 216], all the -dimensional nilpotent Lie algebras [33, p. 646], all the -dimensional nilpotent Malcev algebras [40] and some others.

**The goal of the research.** The goal of this research is to continue the study of Zinbiel algebras and assosymmetric algebras with respect to commutators. Specifically, the research aims to study homomorphic images of free special Tortkara algebras using Lie elements in the free Zinbiel algebra and provide an answer to a question previously posed in [18, p. 70]. An analogy of the classical Cohn’s theorem in Jordan algebras for free special Tortkara algebras is also obtained. Furthermore, the research aims to generalize some properties of associative algebras to assosymmetric algebras related to Lie ideals. The final part of the research is devoted to the algebraic classification of nilpotent assosymmetric algebras, developing a unified algorithm using Wolfram Mathematica code to reduce the computational parts of the classification problem for finite dimensional nilpotent algebras and demonstrate it with a new classification of nilpotent assosymmetric algebras [41 – 43].

**General methodology of the research.** We use methods of structural and combinatorial theory of free Zinbiel and assosymmetric algebras. We study the basic methods of constructing central extensions of nonassociative algebras. We obtain the algebraic classification of small dimensional nilpotent assosymmetric algebras by the Skjelbred-Sund classification method [43, p. 154].

**Scientific novelty.** The main results of the first part of the dissertation are as follows:

- The criteria for determining Lie and Jordan elements in a free Zinbiel algebra is obtained;

- A basis for a free special Tortkara algebra is described;

- An exceptional homomorphic image of a free special Tortkara algebra with three generators is constructed;

- An analogue of Cohn’s theorem for a free special Tortkara algebra is proved. That is, the speciality of any homomorphic image of a free special Tortkara algebra with two generators is proved;

- It was proved that there is no special identity with two generators.

For every assosymmetric algebra we form a series of ideals

It is said that is of finite class if for some positive integer . For the minimal integer such that , we call the class of [19, p. 343].

The main results of the second part of the dissertation are as follows:

- It was obtained that if be an assosymmetric algebra of finite class, then is nilpotent of nilpotent index less or equal to the class of ;

- It was proved that if or is odd for every assosymmetric algebra where is the commutator ideal of ;

- The algebraic classification of nilpotent 4-dimensional assosymmetric algebras is obtained;

- The algebraic classification of nilpotent 5- and 6-dimensional assosymmetric algebras with one generator is obtained.

**Theoretical and practical significance.** The theoretical significance of this work lies in advancing the understanding of the structures of algebras and PI-theory. The results obtained in this research can be used to develop the theory of these structures further and to gain a deeper understanding of how they behave. Additionally, the findings can be applied to the study of finite dimensional assosymmetric algebras and free Tortkara and Zinbiel algebras.

In terms of practical significance, the results of this research can be used in various fields that rely on the use of algebras, such as mathematics, physics, and computer science. The results can also be used to improve the methods used to classify algebras and to develop new algorithms for solving problems related to algebras.

**Publications.** During the period of doctoral studies, 7 publications were published in international journals. Scopus and Thomson Reuters index these journals. The main results on the topic of the dissertation were published in the form of articles in peer-reviewed journals [41 – 44]. There are 3 articles that are not related to the topic of the dissertation [45 – 47]. Moreover, the authors of the published work [43] were awarded the Leader of Science Web of Science Award 2020 in the category of the most cited author from Kazakhstan by Clarivate Analytics.

The results of this dissertation were reported at:

- “Annual Scientific April Conference”, Institute of Mathematics and Mathematical Modeling (2022, Almaty, Kazakhstan);

- the scientific seminar of the Institute of Mathematics named after V.I. Romanovsky (2021, Tashkent, Uzbekistan);

- the scientific seminar of Astana IT University (2021, Nur-Sultan, Kazakhstan).

- III International Workshop on “Non-Associative Algebras in Malaga”, University of Malaga (2020, Malaga, Spain);

- the regular scientific seminar of the School of Mathematics and Cybernetics of the Kazakh-British Technical University (2019-2021, Almaty, Kazakhstan);

- the algebraic seminar of the Faculty of Engineering and Natural Sciences of the Suleyman Demirel University (2019-2021, Kaskelen, Kazakhstan);

**The structure and scope of the thesis.**

The dissertation consists of an introduction, three chapters, a conclusion, a list of references, and an appendix, for a total of 86 pages.

In Chapter 1, the fundamental notions and properties of nonassociative algebras are defined and recalled. Additionally, known results about specific types of nonassociative algebras, such as Jordan, Zinbiel, Tortkara, and assosymmetric algebras, are presented.

The next chapter is devoted to the study of free Zinbiel algebras over a commutator. The first section of this chapter is dedicated to obtaining the main lemmas, which are subsequently used to prove the main theorems of the chapter.

Let and be a free Zinbiel algebra on . Define a linear map on base elements as follows

where

The first main result of this chapter is the following theorem, which gives us the Lie criterion for elements in a free Zinbiel algebra:

**Theorem 2.2.8** *Let be a Zinbiel element of Then is a Lie element if and only if*

The next theorem demonstrates a base of free special Tortkara algebra:

**Theorem 2.2.9** *The set of skew-right-commutative elements , where forms base of*

Moreover, we show that every identity with two generators is a consequence of anticommutativity and Tortkara identities:

**Theorem 2.4.2** *The free Tortkara algebra is special.*

The next theorem is an analogue of Cohn’s theorem on the speciality of homomorphic images of the free special Jordan algebras with two generators [7, p. 261]. For a free special Tortkara algebra with three generators, we have an exceptional homomorphic image, we show this by constructing a counter-example.

**Theorem 2.5.1** *Any homomorphic image of a free special Tortkara algebra with two generators is special. For the three generators case, this statement is not true: a homomorphic image of special Tortkara algebra with three generators might be non-special.*

The results of Chapter 2 were published in [44].

The third chapter of this dissertation focuses on the study of assosymmetric algebras of finite class and commutator ideals of assosymmetric algebras. The main objective of the initial section is to examine the properties of assosymmetric algebras of finite class and demonstrate that they possess similar characteristics to associative algebras of finite class. The results obtained in this section can be used to further develop the theory of assosymmetric algebras and gain a deeper understanding of how they behave. The last part of this chapter is devoted to the algebraic classification of nilpotent assosymmetric algebras, where a unified algorithm is developed using Wolfram Mathematica code to reduce the computational parts of the classification problem.

We obtain an analogue of Jennings’ result from [19, p. 346] for assosymmetric algebras:

**Theorem 3.1.6** *Let be an assosymmetric algebra of finite class. Then is nilpotent of nilpotent index less than or equal to the class of*

We have a generalization of of Corollary 1.4 in [48] for associative algebras.

**Theorem 3.1.8** *Let be an assosymmetric algebra. Then we have the following if or is odd.*

The results of this section were published in [49].

The final section of this chapter concentrates on the algebraic classification of finite dimensional nilpotent assosymmetric algebras. The section starts with an overview of the necessary background information to apply the well-known Skjelbred-Sund classification method and the algorithms we follow in writing the code. We provide new results to illustrate our unified symbolic computational approach.

**Theorem 3.2.5** *Let be a nonzero -dimensional complex nilpotent assosymmetric algebra. Then, is isomorphic to one of the algebras listed in Table A.1 in Appendix A.*

Regarding the and -dimensional nilpotent assosymmetric algebras, applying the same algorithm we have the following theorem:

**Theorem 3.2.7** *Let be a - or -dimensional complex one-generated nilpotent assosymmetric algebra, then is isomorphic to an algebra from Table A.3 or Table A.5 in Appendix A.*

The results of this section were published in [41 – 43].

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# 1 THEORETICAL FRAMEWORK

This chapter presents the main notions, definitions, and theorems that we need in our theorems and in their proofs. In addition, in the last sections, we recall some known results in the theory of Jordan, Zinbiel, Tortkara and assosymmetric algebras. It is based on books [50 – 52].

## 1.1 Basic properties of algebras

Let be a vector space over a field with given bilinear mapping (usually called multiplication) on such that

where for and for all . Then is called an algebra over field . The multiplication is often abbreviated by . The dimension of the algebra is its dimension as a vector space. The algebra is finite-dimensional if is a finite-dimensional vector space.

Let be a set. The free nonassociative algebra over a field from the set of generators is defined by the following universal property: For any algebra A, any mapping can be uniquely extended to the algebra homomorphism The cardinality of the set X is called the rank of We can construct the free algebra using a set of nonassociative words of the set X which is defined inductively:

for No other sequences of the elements from and brackets are not contained in

Let us define the multiplication on as

Now we consider to be a set of formal sums

and extend the operation of multiplication defined on to by the rule

where and We obtain the free nonassociative algebra over a field from the set of generators . The elements of are called *nonassociative polynomials* of (noncommutative) variables from the set

A *monomial* is a polynomial with only one term, which can be written in the form of , where is a scalar from the field and is a polynomial in the variables . The *degree* of a monomial is the length of the word . The degree of a polynomial is the highest degree of its monomials A monomial is said to have a *multi-degree* if it contains exactly times, A *homogeneous* *polynomial* is a polynomial where all monomials have the same multi-degree.

A homogeneous polynomial is called *multilinear* if it is linear in any of its variables (it is homogeneous of multidegree

The linearization of homogeneous polynomials is useful in the study of identities of algebras and in the study of varieties. The process of the linearization is described in detail e.g. in [51, p. 24].

Let be an algebra over a field with multiplication The multiplication is often abbreviated by A nonassociative polynomial is called an *identity* of the algebra if for any where We say that satisfies the identity or that the identity is valid in

For example, an algebra is *commutative* if it satisfies the identity for all The algebra is called *noncommutative* if it is not commutative.

An *associative* algebra is an algebra with identity

An algebra is *nonassociative* if the above identity is not satisfied.

An algebra is *assosymmetric* algebra if it is defined by the following identities:

where

A nonassociative algebra with identity

(1)

is called *(right)-Zinbiel* algebra. Such algebras are called dual of Leibniz or Zinbiel (read Leibniz in reverse order) algebras. Zinbiel algebras were introduced by J-L.Loday in [53].

An algebra A is *anticommutative* if it satisfies the identity

(2)

for all This implies that and the converse holds in characteristic .

The Jacobian in an anticommutative algebra is defined by

A *Lie* algebra is an anticommutative algebra satisfying the Jacobi identity

for all

Anticommuative algebra with identity

(3)

is called  *Tortkara* algebra [1]. If the characteristic of the field is different from two, then the identity has the following multilinear form

(4)

An algebra is called Jordan algebra, if it is a commutative algebra with the following identity

(5)

for all [52].

A *subalgebra* of an algebra is a closed under multiplication: (i.e. for any the product belongs to ). A (two-sided) ideal of an algebra is a subalgebra closed under multiplication by , i.e.

Ideals and of the algebra are called improper ideals. The theory of nonassociative algebras defined two subsets of which do behave associatively: The nucleus (or the associative center) of an algebra is the set of elements which associate with every pair of elements in the sense that That is

The center of an algebra is the set of all elements which commute and associate with all elements in . That is

Note that is an associative and is a commutative and associative subalgebra of A. Moreover

A homomorphism of algebras is a homomorphism of vector spaces (i.e. a linear mapping) which saves multiplication,

for each

The set

is a *kernel* and a homomorphic *image* of of the homomorphism is the set

If is injective homomorphism then we say that an algebra is embedded in . A homomorphism of algebras which is bijective is called isomorphism (of algebras). An endomorphism is a homomorphism of algebras If is an ideal of then the mapping such that is called a natural (or canonical) homomorphism of algebras.

**Theorem 1.1.1 (Fundamental theorem of homomorphism for algebras [50, p. 9])** *Let , be algebras. Let be an ideal of be a homomorphism of algebras and the natural homomorphism. Then there is a unique homomorphism such that Furthermore, is an isomorphism if and only if is a surjective homomorphism and*

**Theorem 1.1.2 (Isomorphism theorem [50, p. 10])**  *If is a homomorphism of algebras over the field then*

*If and are ideals of the algebra with , then*

(iii) *If is a subalgebra of and is an ideal of , then is an ideal of and*

An algebra is called *nilpotent* if for some where are defined by

An algebra is called *solvable* if for some where are defined by

The minimal such is called the *index of nilpotency* *(index of solvability*, respectively) of the algebra . Clearly, any nilpotent algebra is solvable. The concepts of solvability and nilpotency are equivalent for associative algebras:

## 1.2 Variety of algebras

Let be a set of polynomials from Then the class of all algebras satisfying this set of identities is called the *variety of algebras* over the field defined by the set of identities A *subvariety* is a subset of a variety which is itself a variety. Algebras from the variety are called shortly - algebras. The variety consisting of only the zero algebra is called trivial. The variety is called *homogeneous* if, for every identity satisfied in the variety , all the homogeneous components of are also satisfied in .

**Proposition 1.2.1 [51, p. 17]** *Every variety of algebras over an infinite field is homogeneous*.

**Proposition 1.2.2 ([51 p. 25])**  *Over a field of characteristic zero any homogeneous identity is equivalent to a multilinear identity*.

**Proposition 1.2.3 ([54], see also [55, p. 181])** *Over a field of characteristic zero any variety can be defined by multilinear identities*.

**Example 1.2.4** *The variety of associative algebras is defined by one identity*

**Example 1.2.5** *The variety of assosymmetric algebras is defined by the identities*

**Example 1.2.6** *The variety of Zinbiel algebras is defined by the identity*

Let us denote by

an algebra A satisfies all the identities from

the variety of algebras defined by the set Notice that where denotes the ideal generated by Similarly, denote

for all

the set of all the identities that are satisfied in and

Any variety of algebras is closed under homomorphisms, subalgebras, and direct products by results in [55, 56]. And to decide whether a class of algebras forms a variety is used following Birkhoff’s (or HSP) theorem.

**Theorem 1.2.7 (Birkhoff theorem [55, p. 172])** *A class of algebras form a variety if and only if is closed under Homomorphisms, Subalgebras and direct Products (HSP).*

The algebra is called a -free (relatively free or free in the variety ) with the set of generators X, if for any algebra every mapping

can be uniquely extended to a homomorphism of the algebras

The -free algebra is not free in general but only in the variety , i.e. it satisfies identities (and their consequences) that define the variety The construction of the -free algebra is explained by the following theorem:

**Theorem 1.2.8 [51, p. 13]** *Let be a nontrivial variety with the system of defining identities I. Then for any set the natural homomorphism is injective and the quotient algebra is free in the variety V with the free set of generators . Any two free algebras in with equivalent sets of free generators are isomorphic.*

The commutator in algebra is the bilinear function

The *minus algebra* of algebra is the algebra with the same underlying vector space as but with the multiplication .

The Jordan product (or anticommutator) in algebra is the bilinear function

The *plus algebra* of algebra over a field is the algebra with the same underlying vector space as but with as the multiplication.

## 1.3 Special Jordan algebras

The theory of associative algebras and Jordan algebras is particularly important in the study of algebra. Let be a variety of associative algebras. Let class of algebras of types Well known, that any algebra in satisfies the Jordan identity Jordan algebra is *special* if it is isomorphic to a subalgebra of the algebra for some associative algebra . Otherwise, it is *exceptional*.

The study of free algebras is very important. Here we shall need two of them: the  *free (unital) associative algebra* and the *free (unital) special Jordan algebra* which is the Jordan subalgebra of generated by and 1.

A fundamental result in the theory of free special Jordan algebras is Proposition 1.3.1, also known as the universal property of free special Jordan algebras.

**Proposition 1.3.1 (The universal property of Free special Jordan algebras [51, p. 76])** *Let be a special Jordan algebra with a unit If there is a unique homomorphism suc h that and*

Let be a set and be a free associative algebra generated by . A polynomial in is called Jordan element of if it can be expressed by elements of in terms of anticommutators. There is still no criterion that determines all Jordan elements in This problem is solved only for some subspaces of the space of all Jordan elements.

**Lemma 1.3.2 (P. Cohn [7, p. 255])** *Let be an ideal of free special Jordan algebra and is an ideal of free special associative algebras generated by the set Then is a special Jordan algebra if and only if*

Let us define in the involution by

and the element in is called *reversible* if The is called a reversible element.

**Theorem 1.3.3 (P. Cohn [7, p. 257])** *Every reversible element of can be expressed as a Jordan polynomial in generators and the elements*

where and

The expression is called a *tetrad*.

**Corollary 1.3.4 (P. Cohn [7, p. 259])**  *If the number of generators is less than four then the free special Jordan algebra coincide with the space of reversible elements in*

**Theorem 1.3.5 (P. Cohn [7, p. 262])**  *Let is free special Jordan algebra generated by and be the ideal in generated by element Then is exceptional.*

## 1.4 Zinbiel algebras

In some papers Zinbiel algebras are called *dual Leibniz*, *chronological* or *pre-commutative algebras* [18, p. 145], [57 – 59].

Let be a set. Let be the free Zinbiel algebra generated on For denote by a left-bracketed element In [53, p. 190] it was proved that the following set of elements

forms a base of the free Zinbiel algebra

Now we present some results from [60], about the Zinbiel algebras. Recall that an algebra is Zinbiel, if for any relations are satisfied.

**Theorem 1.4.1 [60, p. 197]** *Let be an algebraically closed field of characteristic Then every finite-dimensional Zinbiel algebra is solvable.*

The next theorem gives information about solvable Zinbiel algebras.

**Theorem 1.4.2 [60, p. 197]** *Let be a field of characteristic and be a solvable Zinbiel algebra with solvability length If or then is a nil-algebra with nil-index no greater than Conversely, if is a Zinbiel nil-algebra with nil-index and if or then is solvable with solvability length*

**Theorem 1.4.3 [60, p. 197]** *Let be a field of characteristic Every Zinbiel nil-algebra is nilpotent. If is a nil-algebra with nil-index then the nilpotency index of is no greater than*

**Corollary 1.4.4 [60, p. 197]** *Every finite-dimensional, simple Zinbiel algebra over an algebraically closed field of characteristic is isomorphic to the algebra with trivial multiplication.*

**Corollary 1.4.5 [60, p. 197]** *Every finite-dimensional Zinbiel algebra over the field of complex numbers is nilpotent (and, hence solvable and nil). If then every finite-dimensional Zinbiel algebra over an algebraically closed field of dimension less than and characteristic is nilpotent (and hence solvable and nil).*

Let

be the center of

**Corollary 1.4.6 [60, p. 197]** *Let be a finite-dimensional Zinbiel algebra over the field of complex numbers of dimension Then there exists such that the product of any elements of in any type of bracketing is equal to Moreover, has the nontrivial center The same is true for any finite-dimensional Zinbiel algebra over a field of characteristic if*

Let be a variety of Zinbiel algebras. Define and as classes of algebras of types and defined on the space by  *anti-commutator* and  *commutator* , respectively. Any algebra in is commutative and associative [53, p. 191]. It was proved in [1] that any algebra in satisfies the identity

where is the Jacobi of elements . A  *Tortkara* algebra is defined as anticommutative algebra that satisfies Tortkara identity.

**Theorem 1.4.7 [1, p. 3911]** *For any Zinbiel algebra its Lie algebra where satisfies the identity Tortkara. Any identity of degree 3 of the category follows from the anticommutative identity. Any identity of degree 4 for the category follows from the identities anticommutativity and Tortkara.*

In [1] obtained that the algebra , where

is not Zinbiel algebra, but the corresponding algebra under a commutator satisfies the Tortkara identity.

**Theorem 1.4.8 [1, p. 3911]** *The algebra satisfies the right-symmetry identity*

*and the identity of degree four*

*where Then, its minus algebra satisfies the Tortkara identity.*

An identity is called *special* if it holds in any homomorphic image of a special Tortkara algebra but does not hold in all Tortkara algebras. We still do not know whether there exists a special identity.

P. Kolesnikov proved that the classes neither nor are closed under the operation of taking homomorphic images and are therefore not variety [18, p. 153-154]. In the commutator case, P.S Kolesnikov constructed an example of special Tortkara algebra on four generators which can not be embedded into the commutator algebra of a Zinbiel algebra.

**Theorem 1.4.9 [18, p. 153]** *The algebra has an exceptional homomorphic image.*

**Corollary 1.4.10 [18, p. 153]** *The class of all special Tortkara algebras is not variety.*

In the anti-commutator case, he showed that there is a homomorphic image of a free anti-commutator algebra on one generator that is not embedded into the anti-commutator algebra of a Zinbiel algebra.

**Theorem 1.4.11 [18, p. 154]** *The algebra has an exceptional homomorphic image.*

## 1.5 Assosymmetric algebras

In this section, we present some results about assosymmetric algebras. Assosymmetric algebras are a type of nonassociative algebra that are of interest in the study of identities of algebras and varieties.

**Theorem 1.5.1 (E. Kleinfeld [22, p. 983])** *If is an assosymmetric algebra without ideals such that then is associative, provided the characteristic of is different from and*

Let be a free assosymmetric algebra on Denote by or a *left-normed element* for The element

is called *ordered expression*, where and we have order and In addition, it is known that the set of left-normed and ordered expression elements forms a basis for the free assosymmetric algebra over a field of characteristic This was shown in [27**,** p. 312], where a multiplication rule of the base elements was also given. These multiplication rules are summarized in Proposition 1.5.2, which includes equations (6) - (8) and (9).

**Proposition 1.5.2 [27, p. 312]**

(6)

(7)

(8)

(9)

Furthermore, in [28, p. 32], it was proved that each solvable assosymmetric algebra of characteristic different from 2 and 3 is nilpotent.

**Theorem 1.5.3 (D. Pokrass and D. Rodabaugh [28, p. 32])** *Let A be a solvable assosymmetric ring of characteristic Then A is nilpotent.*

This result, stated in Theorem 1.5.3 by D. Pokrass and D. Rodabaugh, gives us the motivation to classify low-dimensional nilpotent assosymmetric algebras over a field of characteristic 0. To accomplish this task, we will first analyze the properties of homogeneous polynomials in assosymmetric algebras and employ the Skjelbred and Sund method to classify them.

# 2 SPECIAL TORTKARA ALGEBRAS

In this chapter, we consider Zinbiel algebras under commutators and anticommutators. We give criteria for Lie and Jordan elements in a free Zinbiel algebra and by using criteria we obtain the main results of this chapter. All results of this chapter is published in [44].

## 2.1 Definitions and notations

We recall definition of a linear map on base elements as follows

where

For set

Since it is clear that

Let be an integer and let be set of sequences such that For set

We call elements of the form , where  *skew-right-commutative* or shortly  *skew-rcom* elements of

For instance, for we have

Recall the definition of the Lie element. We say that for in a free Zinbiel algebra is *Lie element* if it can be shown as a linear combination of words on *X* under the product Similarly, an element is referred to as a *Jordan element* if it can be expressed as a linear combination of words on *X* using the product Next, we define as a free special Tortkara algebra on under the commutator, i.e., subalgebra of generated by Furthermore, is defined as a subalgebra of the generated by

Define  *Dynkin map* on base elements as follows

## 2.2 Lie elements in a free Zinbiel algebra

In this section, we give Lie criterion for elements in a free Zinbiel algebra.

### 2.2.1 Shuffle permutations

Let be set of shuffle permutations, i.e.,

.

For any positive of integers and denote by set of sequences constructed by shuffle permutations by changing to if and to if

For example,

The following proposition, which was established in [53] for free left-Zinbiel algebras, can also be derived for free right-Zinbiel algebras..

**Proposition 2.2.1.** **(Loday [53, p. 192])**

*Proof.* The validity of the formula can be established through induction on *n = p + q* and the use of identity (1).

The shuffle product of two base elements and in the free Zinbiel algebra is defined as follows:

**Proposition 2.2.2**  *The shuffle product on has the following properties:*

1. *the shuffle product is commutative and associative*

*for any*

For example,

*Proof.* All these properties follow Proposition 2.2.1 and the definition of the shuffle product.

### 2.2.2 Products of skew-right-commutative elements

In the following lemma, we define the product of skew-right-commutative elements in the Zinbiel algebra.

**Lemma 2.2.3**  *Zinbiel product of skew-right-commutative elements can be presented as follows*

*and for*

*Proof.* Let us prove the first part of the lemma,

(by part  **b** of Proposition 2.2.2)

(by the definitions of shuffle product and skew-rcom elements)

Let

(by part  **b** of Proposition 2.2.2)

(by part  **c** of Proposition 2.2.2)

(by definition of skew-right-commutative elements we obtain)

This completes the proof.

In the next lemma we show that the commutator product of skew-rcom element by generator is a linear combination of skew-rcom elements.

**Lemma 2.2.4**

*Proof.* Since, we start proof of lemma from

*=*

Now suppose Then

(by part  **b** of Proposition 2.2.2)

(by part  **c** of Proposition 2.2.2)

(by the definition of skew-right-commutative elements we obtain)

This completes the proof.

**Lemma 2.2.5**  *The commutator of skew-right-commutative elements is a linear combination of skew-right-commutative elements.*

*Proof.* If skew-right-commutative elements have degree 2, then a straightforward calculation shows that

The proof of the assertion is presented below under the assumption that the degree of skew-right commutative elements is at least 3. The case when one of the elements has degree 2 is established using similar way. So,

(by Lemma 2.2.3)

(by part  **a** of Proposition 2.2.2)

This completes the proof.

**Lemma 2.2.6**  *If is an element of then*

*Proof.* The proof is achieved by the fact that is generated by the commutator products on and for any and by using Lemma 2.2.4 and Lemma 2.2.5.

Now we prove that any skew-right-commutative element of is a Lie element.

**Lemma 2.2.7**  *Let is an element of with Then*

*Proof.* Write if If then can be expressed as a linear combination of skew-right-commutative elements. In order to prove that , it is enough to show that

(10)

We prove it by induction on

If the proof is straightforward, and if we have:

Assuming that equation (10) holds for elements of degree less than , we have

for any Lie element whose degree is no more than Set and have

for

Hence,

Since the symmetric group is generated by transpositions , for any we have

(11)

By (11) and Lemma 2.4 we have

Also, can be obtained the following

*.*

Consider the sum of the above last two expressions and we have

Thus

In other words,

(12)

Set By (11) and (12) we have

Hence,

and this completes the proof.

Now we are ready to prove the main theorems of the section.

**Theorem 2.2.8**  *Let be a Zinbiel element of Then is a Lie element if and only if*

*Proof.* It follows from Lemma 2.2.6 and Lemma 2.2.7.

**Theorem 2.2.9**  *The set of skew-rcom elements where forms base of*  *Let be the homogenous part of generated by generators where Then*

*where In particlular, the multilinear part of has dimension*

*Proof.* Since a skew-rcom element defined as

a difference of two base elements, any linear combination of skew-rcom elements is trivial, hence they are linearly independent in By Lemma 2.2.6, any element of is a linear combination of skew-rcom elements. So we have proved that the set of skew-rcom elements, generated by set forms a base of

Let us count the number of skew-rcom elements of degree generated by in which occur times, respectively. Consider skew-rcom elements whose last two elements are for Then the number of such type of skew-rcom elements of degree equals

where Hence

If for all then and we obtain

This completes the proof.

**Corollary 2.2.10**  *Let If and are Lie, then and are Lie.*

*Proof.* We present a proof of our Corollary for The case can be established in a similar way.

Let and suppose . Then by Theorem 2.2.9

We have

Let and The cases, when at least one of and is equal to two, can be easily proved. Suppose Set and

(by part  **b** of Proposition 2.2.2)

(by part  **c** of Proposition 2.2.2)

(by the definition of skew-right-commutative elements we obtain)

By similar way one can have

So

(by part  **a** of Proposition 2.2.2)

Hence by Theorem 2.2.8 we have and this completes the proof.

## 2.3 Jordan elements in a free Zinbiel algebra

In this section, we demonstrate the proof of the Jordan criterion for elements in a free Zinbiel algebra We provide an explicit formula for expanding Jordan bracketed elements in a free Zinbiel algebra.

**Lemma 2.3.1**

*Proof.* We prove it by induction on For the base of induction we have Suppose that it is true for Then

(by induction hypothesis)

(by Proposition 2.2.1)

This completes the proof.

**Theorem 2.3.2**  *Let be a homogenous Zinbiel element of degree in Then is a Jordan element if and only if The algebra is isomorphic to polynomial algebra*

*Proof.* Recall that is associative and commutative algebra if is Zinbiel. Any Jordan element in can be written as linear combination of left-normed Jordan monomials in by anti-commutators. Then the proof follows from Lemma 4.1 and definition of the map

Let be a canonical homomorphism from polynomial algebra generated by to defined as Then it is clear that is zero and therefore and are isomorphic. This completes the proof.

Denote by the homogenous part of generated by generators where

**Corollary 2.3.3**  *The dimension of the homogenous part of is equal to*

*Proof.* It is an immediate consequence of Theorem 2.3.2.

## 2.4 Speciality of the free Tortkara algebra with two generators

In this section we prove that the free Tortkara algebra with two generators is special. As a corollary, we obtain the construction of a base of in terms of left-normed elements.

**Lemma 2.4.1** *Let be the n-th homogenous part of Then for any*

*Proof.* Clearly, We write if We prove the statement by induction on degree We have

This is the basis of induction for Suppose that our statement is true for fewer than Let and where and are elements of whose degrees are and respectively, and Now we consider induction on By induction on we may assume that they are left-normed and write

where Suppose and Assume Then by Tortkara identity (4) and induction on we have

Suppose that our statement is true for fewer than We have

(by anticommutativity identity)

(by identity (4))

We note that by base of induction on ,

By induction on we have

Hence, This means that any element incan be written as a linear combination of left-normalized elements.

Let be a free Tortkara algebra generated by a set

**Theorem 2.4.2**  *The free Tortkara algebra is special.*

*Proof.* It is sufficient to show that algebras and are isomorphic. Let be a natural homomorphism from to By Lemma 2.4.1 the vector space is spanned by the set of left-normed elements. We note that number of left-normed elements in two generators is equal to the number of skew-rcom elements in two generators. Suppose that the kernel of is not zero. Then we have a linear combination of skew-rcom elements which is zero in It contradicts to the first part of Theorem 2.2.9. Therefore, This completes the proof.

**Corollary 2.4.3.** *Set of left-normed elements forms a base of*

*Proof.* It is an immediate consequence of Theorem 2.4.2.

## Speciality of homomorphic images of

The next theorem is an analogue of Cohn’s theorem on the speciality of homomorphic images of special Jordan algebras in two generators Lemma 1.3.4[7, p. 255])*Let be an ideal of free special Jordan algebra and is an ideal of free special associative algebras generated by the set Then is a special Jordan algebra if and only if*

.

**Theorem 2.5.1**  *Any homomorphic image of a free special Tortkara algebra with two generators is special. For the three generators case, this statement is not true: a homomorphic image of special Tortkara algebra with three generators might be non-special.*

Let be an ideal of By Cohn’s criterion (Theorem 2.2 of [7, p. 255]) is special if and only if where is the ideal of generated by the set

*Proof of Theorem 2.5.1.* Assume that are generators of the ideal It is clear that if then is special.

Therefore, by Theorem 2.2.8 we can assume that each element has a form for some

Let be a non-zero element of . Then and is a linear combination of left-normed monomials in such that each monomial is linear by at least one generator of . Let be a term of in the linear combination. To prove the statement we consider two cases, depending on what position a generator appear in

*Case 1.* Suppose that generators of appear only in the first positions in Then write all in terms of elements of Since , must have the term with opposite sign. Hence

*Case 2.* Suppose that generators of appear in either -th or -th positions in (a generator of may appear in the first -positions), namely,

и или и

for some If generators of appear in both -th and -th positions of then write one of them in terms of and We also express in terms of and , therefore we can assume that Let us denote by .

Now we show that if is a term of then has the term with opposite sign.

We have

where

,

By Theorem 2.2.8, is Lie, but are not Lie. Since the term should be cancelled and for each term of must have terms or with opposite sign to cancel or have Therefore, must have some terms in which appear in either -th or -th positions. These kind of terms are generated by Then all possibilities of such types are and We have

We see that the element is a term of only and moreover,

Therefore, has the term with opposite sign. Since is a generator of and by Corollary 2.2.3

Hence If is a nonzero term of , then by similar way one can show that must have term .

So we obtain It follows Hence by Cohn’s criterion is special.

Now we show that a homomorphic image of may be not special. Let be an ideal of generated by elements

Consider an element

Then

It follows

One can easily check that there are no so that

Then Hence by Cohn’s criterion, is not special.

**Corollary 2.5.2**  *Any Tortkara algebra with two generators is special.*

*Proof.* It follows from Theorem 2.4.2 and Theorem 2.5.1.

This result is an analogue of Shirshov theorem for Jordan algebras [51, p. 84].

## 2.6 Some remarks and open questions

1. Let be an algebra with multiplication

(13)

is not a Zinbiel algebra. This algebra w was considered in [1]. It was proved that it satisfies the following identities

(14)

where Moreover, it was proved that algebra with respect to commutator is a Tortkara algebra. A question on speciality of was posed.

We show that answer is positive. Let be an algebra with multiplication

Then is a Zinbiel algebra. For multiplication we define commutator Note that So is isomorphic to Hence is special.

**2.** It is shown in [1] that an algebra with identities (14) is not Zinbiel but under the commutator product is Tortkara. What about speciality of these algebras?

**3.** Let be kernel of the natural homomorphism from free Tortkara algebra to free special Tortkara algebra on generators. An element of the ideal is called a -identity. We showed that . Are there -identities for ?

**4.** Is it true the analogue of Lemma 4.1 for generators? Whenever it is valid for generators, it immediately follows speciality of in particular, .

# 3 NILPOTENT ASSOSYMMETRIC ALGEBRAS

In this chapter, we mainly study assosymmetric algebras of finite class and study commutator ideals of assosymmetric algebras. We show that some of the properties for associative algebras also hold for assosymmetric algebras, namely, for such properties associativity is not necessary and can be replaced by left-symmetry and right-symmetry. The results of this chapter were published in [41 – 43, 49].

## 3.1 Commutator ideals of assosymmetric algebras

We begin with some basic facts on Lie-admissible algebras. The results of this section were published in [49].

Let be an arbitrary Lie-admissible algebra over a given field . We define

for all and . For all subspaces , , *D* of , we define

and

where the associator means . We call a space a *Lie ideal* of if we have . Finally, for all subspaces and of , we define

that is, the ideal of generated by . Following the idea of Jennings [19, p. 341], we call the *commutator ideal* of and . We clearly have .

Equipped with the notion of commutator ideals, we are now able to recall the notion of central chains of ideals of a Lie-admissible algebra .

Let

(15)

be a chain of ideals of . Such a chain is called a *central chain of ideals* if we have

(16)

We shall soon see that Novikov algebras, bicommutative algebras and assosymmetric algebras which possess central chains of ideals have special properties; we investigate some of them by considering a particular central chain:

**Definition 3.1.1**  *For every Lie-admissible algebra we form a series of ideals*

(17)

We say that is of *finite class* if for some positive integer . For the minimal integer such that , we call the *class* of , and call

(18)

the *lower central chain* of . To avoid too many repetitions, we shall fix the notation of for all .

With the notations of (16) and (18), it is straightforward to show that by induction on .

The next subsection provides us with a description of commutator ideals of assosymmetric algebras.

### 3.1.1 Assosymmetric algebras of finite class

The aim of this subsection is to study assosymmetric algebras of finite class. Recall that for all in an algebra , the associator means . So in every assosymmetric algebra , we have for all .

It is proved in [22, p. 984] that for all in an assosymmetric algebra , we have

(19)

By the same technique developed in [22, p. 983], we obtain some more identities as follows when or .

**Lemma 3.1.2** *For all in an assosymmetric algebra , we have*

(20)

(21)

(22)

*Proof.* (i) Proof of identity (20). Note that

The proof of identity (20) is completed.

(ii) Proof of identity (21). Following [22, p. 984], we define

Then it is obvious that

(23)

We also note that [22, p. 984] in any algebra we have

(24)

By identity (24), we deduce

Combining this with identity (23), we obtain

(25)

and thus

(26)

Therefore, if , we obtain

The proof of identity (21) is completed.

(iii) Proof of identity (22). If , then we have

Identity (22) follows immediately.

Now we begin to study associators involving Lie ideals of an assosymmetric algebra .

**Lemma 3.1.3**  *Let and be Lie ideals of . Then the following statements are true:*

1. *For all , , ; In particular, is contained in the ideal of generated by ;*
2. *.*

*Proof.* (i) By identity (20), we deduce

The proof is completed.

(ii) Clearly, is a Lie ideal of . It follows that

In particular, is an ideal of if and only if so does .

By (i) , for all , , , we have

It follows that

Therefore, we deduce

The proof is completed.

**Corollary 3.1.4**  *Let be a family of ideals of such that for every . Then for all , for all , we have .*

*Proof.* By Lemma 3.1.3, we have

The proof is completed.

Let

(27)

be a central chain of ideals of . And let be as in Definition 3.1.1. When is assosymmetric, we have the following analogues as those for associative algebras. Again, as the associativity does not hold, new techniques are necessary.

**Lemma 3.1.5**  *Let be an assosymmetric algebra. Then we have , , and . In particular, we have .*

*Proof.* Since , and , it suffices to prove , and .

We use induction on to prove these claims. For , we have , and by Corollary 3.1.4, we obtain

Now we assume . If for some and , then for every , by induction hypothesis, we have

By the Jacobi identity and induction hypothesis, we obtain

We continue to show for every . There are several cases to discuss depending on the characteristic of the field. If , then by identity (19), we have . If , then by identity (21) and by Corollary 3.1.4, we have

If , then by identities (20) and (22) and by the above reasoning, we have

Now we prove for the case when and for some elements and . By the above reasoning and by the right-symmetric identity, we have

By identity (20) and by the above reasoning, we obtain

Finally, by the above reasoning and by the induction hypothesis, for every , we deduce

The proof is completed.

**Theorem 3.1.6**  *Let be an assosymmetric algebra of finite class. Then is nilpotent of nilpotent index less or equal to the class of .*

*Proof.* By Lemma 3.1.5 and by similar reasoning as the proof for Theorem 2.5 in [49], we obtain the description for assosymmetric algebras that generalizes the corresponding result of associative algebras.

### 3.1.2 Products of commutator ideals of assosymmetric algebras

The aim of this subsection is to study products of commutator ideals of an arbitrary assosymmetric algebra over a field such that .

Let be an assosymmetric algebra. We define

We call the *th commutator ideal* of . And the algebra is called *Lie nilpotent* if for some integer . We shall prove that if is odd or is odd, which generalizes the corresponding result [48, p. 300] for associative algebras.

For all , we define

The main difference in the above-mentioned result between associative algebras and assosymmetric algebras is the proof of the following lemma.

**Lemma 3.1.7**  *Let be an assosymmetric algebra over a field such that . For every positive odd integer , we have . Moreover, we have .*

*Proof.* For all , we have . So the second claim follows immediately from the first one. We use induction on to prove the lemma. For , the claim follows immediately by the definition of and by the above reasoning if . Now we assume that is an odd integer such that . For all , it suffices to show if . By assumption, we have , so we have

So in order to show , it suffices to prove . The idea of the proof is to show that is sort of skew-symmetric. More precisely, we shall prove that, if one of lies in then

Since is an assosymmetric algebra, by identity (19), we have

and thus

Let assume that one of lies in . Since , by the induction hypothesis, we obtain that lies in , and thus we deduce

(28)

Similarly, we have

Again, since one of lies in , by the induction hypothesis, we easily obtain that lies in , and thus we deduce

(29)

On the other hand, by identity (28), we have

Interchanging and in the above equation, we obtain

So by the above two Equations and by the Jacobi identity, we deduce

=

Since , we obtain

(30)

Therefore, in the vector space , we have

(31)

(32)

and thus

(33)

Therefore, we deduce

It follows that

Finally, since is Lie-admissible, we obtain

Since , we have . The proof is completed.

We conclude the section with the main result of this subsection, which generalizes the corresponding property of associative algebras. The results also published in [49].

**Theorem 3.1.8**  *Let be an assosymmetric algebra. Then we have if or is odd.*

*Proof.* If or , then clearly we have . Now we assume and . Then by Lemma 3.1.3 (ii) and by identity (19), we have

So it suffices to show if one of and is odd. Since

we may assume that is odd and thus we may assume and . For all , and , by identity (19) and by Lemma 3.1.7, we have

The proof is completed.

We also note that if and are even then in general for associative algebras [61]. Since associative algebras are assosymmetric algebras, we know that if and are even then in general for assosymmetric algebras.

## 3.2 The algebraic classification of nilpotent assosymmetric algebras

The results of this subsection were published in [41 – 43].

Using the classification of all 2-dimensional algebras [62], it is easy to check that all 2-dimensional assosymmetric algebras are associative. The present section presents the algebraic classification of 4-dimensional complex nilpotent assosymmetric algebras.

The variety of assosymmetric algebras is defined by the following identities of right- and left-symmetric:

where

Central extensions are a crucial aspect of our method for classifying assosymmetric nilpotent algebras. The central extensions of Lie and non-Lie algebras have been extensively studied over the years and have been used to classify various types of algebras [36, 63, 64]. Firstly, Skjelbred and Sund devised a method for classifying nilpotent Lie algebras employing central extensions [36]. This method has been utilized to describe all non-Lie central extensions of 4-dimensional Malcev algebras [63], all anticommutative central extensions of 3-dimensional anticommutative algebras [65], and all central extensions of 2-dimensional algebras [66].

### 3.2.1 Method of classification of nilpotent algebras

We now present an adaptation of the Skjellbred-Sund method for the classification of nilpotent asymmetric algebras. This method has been used to classify various varieties of algebras and has been explained in works such as [63, p. 34], [66]. We give only some important definitions. For more detailed information, the interested reader is referred to these sources. We will also use their notation.

Define an assosymmetric algebra over the field of complex numbers and let be a vector space over . The set of all bilinear maps satisfying the condition that

forms a -linear space, which is denoted as These maps are referred as  *cocycles*. Given a linear map from to , we can define cocycle with . *Coboundaries* are defined as the elements of the linear subspace . The  *second cohomology space* is defined to be the quotient space .

Let be the automorphism group of the assosymmetric algebra and let . Every defines , with . It is easily checked that acts on , and that is invariant under the action of So, we have that acts on .

Let be an assosymmetric algebra of dimension over , a -vector space of dimension and a cocycle, and consider the direct sum with the bilinear product “ ” defined by for all . It is straightforward that is an assosymmetric algebra if and only if ; it is called an - *dimensional central extension* of by .

We also call the set the  *annihilator* of . We recall that the  *annihilator* of an algebra is defined as the ideal . Observe that .

**Definition 3.2.1 [63, p. 35]** *Let be an algebra and be a subspace of . If then is called an annihilator component of .*

**Definition 3.2.2 [63, p. 35]** *A central extension of an algebra without annihilator component is called a non-split central extension.*

The following result is fundamental for the classification method. For the proof, we refer the reader to Lemma 5 in [63, p. 35].

**Lemma 3.2.3** *Let be an -dimensional assosymmetric algebra such that . Then there exists, up to isomorphism, a unique -dimensional assosymmetric algebra and a bilinear map with , where is a vector space of dimension m, such that and .*

Now, we seek a condition on the cocycles to know when two -central extensions are isomorphic. Let us fix a basis of , and . Then can be uniquely written as , where . It holds that if and only if all , and it also holds that . Furthermore, if , then has an annihilator component if and only if are linearly dependent in (see [63, Lemma 13]).

Recall that, given a finite-dimensional vector space over , the  *Grassmannian* is the set of all -dimensional linear subspaces of . Let be the Grassmannian of subspaces of dimension in . For and , define . It holds that , and this induces an action of on . We denote the orbit of under this action by . Let

Similarly to Lemma 15 in [63, p. 41], in case , it holds that

and therefore the set

is well defined, and it is also stable under the action of (see Lemma 16 in [63, p. 41]). Now, let be an -dimensional linear space, and let us denote by the set of all non-split -dimensional central extensions of by . We can write

Finally, we have main lemma, which can be proved as Lemma 17 in [63].

**Lemma 3.2.4 [63, p. 41]** *Let . Suppose and . Then the assosymmetric algebras and are isomorphic if and only if*

Then, it exists a bijective correspondence between the set of -orbits on and the set of isomorphism classes of . Consequently, we have a procedure that allows us, given an assosymmetric algebra of dimension , to construct all non-split central extensions of .

Let be an assosymmetric algebra of dimension . Then:

1. Compute base for

2. Compute base for and

3. Compute

4. Compute base for and

5. Compute -orbits on

6. Construct a new finite-dimensional nilpotent assosymmetric algebra associated with a representative of each orbit.

Let be an assosymmetric algebra and fix a basis . We define the bilinear form by . Then the set is a basis for the linear space of the bilinear forms on , and in particular, every can be uniquely written as , where .

We now describe algorithms to handle steps from 1 to 4. The remaining two steps are worked out by hand.

Let be an algebra with basis We use the following notations: is the bilinear form such that

(34)

The set is the basis of Every can be uniquely written as where

Now, we give algorithms to compute the above-mentioned steps. The first algorithm shows how to compute given the dimension, the product rule, and the polynomial identities. It amounts to defining the symbolic equations and calling the symbolic solver from the relevant programming language.

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Автоматически созданное описание

Figure 1 – Computation the basis for the

The next algorithm uses outcomes of Algorithm 1 (Figure 1) together with the same inputs. In this case we aim to compute bases for and . It does not require any tricks to obtain a basis for but simply write them down manually from the given polynomial identities. In terms of coding, this means asking the programming language to read the coefficients of polynomial expressions. As for the second part, we recall that and . Thus, the problem of finding a basis for is equivalent to Изображение выглядит как текст

Автоматически созданное описаниеcompleting the basis of given the basis of .

Figure 2 – Сomputation the bases for and

Computing the , Algorithm 3 (Figure 3), is one of the main steps in the above-described method and the one with a large computational cost. We may represent an automorphism with an invertible square matrix that respects the bilinear product rule. This requires defining a symbolic matrix and defining a system of symbolic equations and finally calling the solve function.

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Автоматически созданное описание

Figure 3 – Finding the automorphism group

The next three algorithms are allocated for step 4 to compute annihilators. The Algorithm 4 (Figure 4) computes the action of the automorphism group on and uses outcomes of Algorithm 2 (Figure 2) and Algorithm 3 (Figure 3). Action of automorphism group defined by where and is matrix form of

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Figure 4 – Action of the automorphism group on

Next algorithm uses Algorithm 5 (Figure 5) to compute bases for . Again one needs to define the system of polynomial equations and call the solver.

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Figure 5 – Finding basis of annihilator

Finally, the last algorithm below (Figure 6) uses outcome of Algorithm 5 (Figure 5) and gives conditions of

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Автоматически созданное описание

Figure 6 – Intersection of and

### 3.2.2 The central extensions of low dimensional nilpotent assosymmetric algebras.

We distinguish two main classes of assosymmetric algebras: the “pure” and the “non-pure” ones. By the non-pure ones, we mean those satisfying the identities and ; the pure ones are the rest.

These “trivial” algebras can be considered in many varieties of algebras defined by polynomial identities of degree 3 (associative, Leibniz, Zinbiel, etc.), and they can be expressed as central extensions of suitable algebras with zero product. Those with dimension are already classified: the list of the non-anticommutative ones can be found in [67], and there is only one nilpotent and anticommutative algebra.



**Theorem 3.2.5**  *Let be a nonzero -dimensional complex nilpotent “pure” assosymmetric algebra. Then, is isomorphic to one of the algebras listed in Table A.1 in Appendix A.*

**Remark 3.2.6** *Let be a -dimensional nilpotent non-associative assosymmetric algebra. Then is isomorphic to one algebra from the following list*

*Proof of Theorem 3.2.5.* There are no nontrivial 1-dimensional nilpotent assosymmetric algebras, and there is only one nontrivial -dimensional nilpotent assosymmetric algebra (namely, the non-split central extension of the -dimensional algebra with zero product):

From this algebra, we construct the -dimensional nilpotent assosymmetric algebra Also, the reference [66, p. 10] gives the description of all central extensions of and . Choosing the assosymmetric algebras between them, we have the classification of all non-split -dimensional nilpotent assosymmetric algebras:

Now we consider -dimensional central extensions of 3-dimensional nilpotent assosymmetric algebras. In the following table, we give the description of the second cohomology space of 3-dimensional nilpotent assosymmetric algebras.

Table 1 – the cohomology space of 3-dimensional nilpotent assosymmetric algebras

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**Remark 3.2.7** *From the description of the cocycles of the algebras , and , it follows that the 1-dimensional central extensions of these algebras are 2-dimensional central extensions of 2-dimensional nilpotent assosymmetric algebras.* Thanks to [66, p. 18-22] we have the description of all non-split 2-dimensional central extensions of 2-dimensional nilpotent assosymmetric algebras:

Then, in the following we study the central extensions of the other algebras.

1. **Central extensions of**  Since the second cohomology spaces and automorphism groups of and (from [68]) coincide, these algebras have the same central extensions. Therefore, from [68, p. 19] we have all the new 4-dimensional nilpotent assosymmetric algebras constructed from :

The multiplication tables of these algebras can be found in Appendix A.

1. **Central extensions of**  Let us use the following notations:

The automorphism group of consists of invertible matrices of the form

Since

we have that the action of on the subspace is given by where

The element gives a central extension of a -dimensional algebra. From here, we have the following new cases:

1. Then

(a) If then choosing we have the representative

(b) If then choosing we have the representative

2. and Then by chossing we have the case

3. and Then where Now we have following cases.

1. If then by choosing

we have the representative

1. If then by choosing

we have the representative

Now, we have the following new algebras constructed from

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1. **Central extensions of**  Let us use the following notations:

The automorphism group of consists of invertible matrices of the form

Since

we have that the action of on the subspace is given by where

It is straightforward that the elements lead to central extensions of -dimensional algebras. The new cases are following:

1. Choosing , we have the representative .

2. Choosing , we have the representative .

We have the following new algebras constructed from

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1. **Central extensions of**  Let us use the following notations:

The automorphism group of consists of invertible matrices of the form

Since

we have that the action of on the subspace is given by where

The element gives a central extension of a -dimensional algebra, then we will consider only cases with We find the following new cases:

1. then choosing and , we have the representative .

2. or then:

(a) if , then choosing , we have the representative .

(b) if , then choosing , we have the representative .

Now we have all the new -dimensional nilpotent assosymmetric algebras constructed from , (see Table A.1 in Appendix A).

Summarizing above results the Theorem 3.2.6is proved

Another main result of the present section is the following theorem:

**Theorem 3.2.8**  *Let be a - or -dimensional complex one-generated nilpotent assosymmetric algebra, then is isomorphic to an algebra from the Table A.3 or Table A.5 in Appendix A.*

From Theorem 3.2.5 we have a description of all -, - and -dimensional one-generated nilpotent assosymmetric algebras:

Table 2 – 2-, 3- and -dimensional one-generated nilpotent assosymmetric algebras

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#### *Proof Theorem 3.2.8.* We consider 2-dimensional central extensions of 3-dimensional one-generated algebras.The second cohomology spaces of algebras given in [43]. Therefore, 2-dimensional central extensions of these algebras gives the following two algebras:

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**Remark 3.2.9** *Extensions of the algebras and give algebras with -dimensional annihilator. Then, in the following subsections we study the central extensions of the other algebras.*

All multiplication tables of 4-dimensional one-generated nilpotent assosymmetric algebras is given in Table A.1 (see, Appendix A). All relevant details about coboundaries, cocycles, and second cohomology spaces for five-dimensional one-generated nilpotent assosymmetric algebras were obtained using the code specified in [41], and can be found in Table A.2 (see, Appendix A).

1. **Central extensions of**  Let us use the following notations:

The automorphism group of consists of invertible matrices of the form

Since

we have that the action of on the subspace is given by where

For -dimensional central extensions we have the following new cases:

1. If then we have the representative

2. If then we have the representative

3. If then we have the representative

From here, we have new -dimensional one generated assosymmetric algebras constructed from

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For -dimensional central extensions we consider the vector space generated by the following two cocycles

Here we have the following cases:

1. If then we have the representative

2. If then we have the representative

3. If then we have the representative

We have the following new -dimensional one-generated nilpotent assosymmetric algebras constructed from , , (see Table A.5 in Appendix A).

1. **Central extensions of**  Let us use the following notations:

The automorphism group of consists of invertible matrices of the form

Since

we have that the action of on the subspace is given by where

For -dimensional central extensionsnote that if then we obtain algebras with 2-dimensional annihilator. Therefore, we have two representatives and depending on whether or not.

We have the following new -dimensional nilpotent assosymmetric algebras constructed from and (see Table A.1 in Appendix A).

For -dimensional central extensions we have only one new -dimensional nilpotent assosymmetric algebras constructed from

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Summarizing results of the previous sections, we have the first part of Theorem

All multiplication tables of -dimensional one-generated nilpotent assosymmetric algebras is given in Table A.3 (see, Appendix A). All necessary information about coboundaries, cocycles and second cohomology spaces of -dimensional one-generated nilpotent assosymmetric algebras were calculated by the code in [41] and given in Table A.4 (see, Appendix A).

**Remark 3.2.11** *Extensions of the algebras and give algebras with -dimensional annihilator. Then, in the following subsections we study the central extensions of the other algebras.*

1. **Central extensions of** Let us use the following notations:

The automorphism group of consists of invertible matrices of the form

Since

we have that the action of on the subspace is given by where

We have the following case:

1. If then choosing we have the representative

2. If we have two representatives and depending on whether or not.

Consequently, we have the following algebras from

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1. **Central extensions of**  Here we will consider the special cases for

The automorphism group of consists of invertible matrices of the form

Let use the following notations:

So,

we have that the action of on the subspace is given by where

We are interested only in the cases with Now we obtain the following cases:

1. For :

(a) If then by choosing and we have the representative

(b) If then by choosing

and

we obtain the representative

From the above cases we have new parametric algebras (see Table A.5 in Appendix A).

2. The condition for gives that is . The base of the second cohomology of this algebra spanned by elements:

Since

we have that the action of on the subspace is given by where

We are interested only in then we have the following cases:

(a) If then for we have the representative

(b) If then also we have two cases:

i. If then and we have the representative

ii. If then and we have the representative

Consequently, we have the following algebras from , (see Table A.1 in Appendix A).

3. The condition gives for that is So, the second cohomology space of spanned by elements:

Since

we have that the action of on the subspace is given by where

Since and choosing we have the representatives and depending on whether or not.

We have the following new -dimensional algebras constructed from (see Table A.5 in Appendix A).

1. **Central extensions of**  If for gives that is . So, the second cohomology space of spanned by elements:

Since

we have that the action of on the subspace is given by where

We are interested in then we have the following cases:

1. If then and we have the representative

2. If then and we have the representative

We have the following new -dimensional algebras constructed from (see Table A.5 in Appendix A).

1. **Central extensions of**  Let us use the following notations:

The automorphism group of consists of invertible matrices of the form

Since

we have that the action of on the subspace is given by where

We suppose that , otherwise obtained algebra gives an algebra with 2-dimensional annihilator. Therefore, consider the following cases:

1. If then we have the representative

2. If then we have the representative

Hence, we have the following new algebras: , (see Table A.5 in Appendix A).

1. **Central extensions of** Let us use the following notations:

The automorphism group of consists of invertible matrices of the form

where Since

we have that the action of on the subspace is given by where

We have only one non-trivial orbit with the representative and get the algebra (see Table A.5 in Appendix A).

Summarizing results we have the second part of Theorem 3.2.7.

# 

# CONCLUSION

In conclusion, the dissertation work has focused on two classical problems in the study of nonassociative algebras, specifically, the study of nonassociative algebras under a commutator and the classification of finitely dimensional nonassociative algebras.

Firstly, a criterion was found for determining the Lie elements in a free Zinbiel algebra. This result is of particular importance as it allows for the identification of Lie elements in a free Zinbiel algebra, which is a fundamental step in understanding the structure and properties of these algebras.

Secondly, a basis for special Tortkara algebras was constructed. This result provides a foundation for further study of these algebras and can be used to develop new techniques and methods. Additionally, it was shown that there exists an exceptional homomorphic image of a free special Tortkara algebra with three generators, and it was proved that any homomorphic image of a free special Tortkara algebra with two generators is special.

Thirdly, it has been proved that there is no special identity with two generators. This result has implications for the study of special identities in special Tortkara algebras and can be used to future research in this area.

Fourthly, an algebraic classification of nilpotent 4-dimensional assosymmetric algebras was constructed. This result provides a comprehensive understanding of the structure and properties of these algebras and can be used to guide further research in this area. Additionally, an algebraic classification of nilpotent 5- and 6-dimensional assosymmetric algebras with one generator was constructed.

Finally, algorithms were provided with code written in Wolfram Mathematica to simplify the computational aspects of the classification problem of nilpotent algebras. The use of Wolfram Mathematica allowed for efficient and accurate computations, and the authors partially used the "solve" function, a symbolic solver built into Wolfram Mathematica, when working with a system of polynomial equations. This is the main function that takes up most of the compilation time. The codes written by the authors in other software, including Matlab and Python, gave the worst results in terms of running time and, in some cases, failed to provide any solutions.

In summary, the results obtained in this dissertation have advanced our understanding of nonassociative algebras and have provided new techniques and methods. These results can be applied to further study of Zinbiel algebras under commutator and can be used in special courses on the theory of free and finite-dimensional algebras. The work also highlights the importance of using appropriate software tools when working with symbolic nonlinear equations and the potential for improving the performance of these tools. Overall, this dissertation work has made a contribution to the field of nonassociative algebras and sets the stage for future research in this area.

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# APPENDIX A

Table A.1 – The list of 4-dimensional nilpotent “pure” assosymmetric algebras.

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**Cohomology spaces of 4-dimensional one-generated assosymmetric algebras.** All multiplication tables of four-dimensional one-generated nilpotent assosymmetric algebras is given in Table 2. In the present table we collect all usefull information about and spaces for all four-dimensional one-generated nilpotent assosymmetric algebras that were counted via code in [41].

#### Table A.2 – Cohomology spaces of 4-dimensional one-generated nilpotent assosym- metric algebras

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Table A.3 – The list of 5-dimensional nilpotent “pure” assosymmetric algebras

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**Cohomology spaces of 5-dimensional one-generated assosymmetric algebras.** All relevant information about coboundaries, cocycles and second cohomology spaces of five-dimensional one-generated nilpotent assosymmetric algebras were calculated by the code in [41] and given in the following table:

Table A.4 – Cohomology spaces of 5-dimensional one-generated nilpotent assosymmetric algebras

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Table A.5 – The list of 6-dimensional nilpotent “pure” assosymmetric algebras

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