Al-Farabi Kazakh National University

and

Ghent University

UDC 517.4/.9(043) Manuscript Copyright

**BAYAN BEKBOLAT**

**Dunkl analysis and application to inverse source problems**

6D060100 – Mathematics

Dissertation for a degree

Doctor of Philosophy (Ph.D.)

Domestic scientific consultant

doctor PhD,

N. Tokmagambetov

Foreign scientific consultant

doctor PhD,

professor

M. Ruzhansky

Republic of Kazakhstan

Almaty, 2024

**CONTENT**

|  |  |
| --- | --- |
| **REGULATORY REFERENCES**.................................................................... | 3 |
| **DESIGNATIONS AND ABBREVIATIONS**.................................................. | 4 |
| **INTRODUCTION**............................................................................................. | 5 |
| **1 PRELIMINARY RESULTS**………………………………………………. | 21 |
| 1.1 Background from elementary function analysis........................................... | 21 |
| 1.2 The Dunkl analysis........................................................................................ | 24 |
| 1.3 Fractional differential operators.................................................................... | 51 |
| **2 PSEUDO-DIFFERENTIAL OPERATORS ASSOCIATED WITH THE DUNKL OPERATOR** ………………………………………………… | 56 |
| 2.1 Pseudo-differential and amplitude operators on Schwartz spaces................ | 57 |
| 2.2 Kernel of pseudo-differential operators…………………………………… | 72 |
| 2.3 Boundedness of pseudo-differential operators generated by the Dunkl operator………………………………………………………………………... | 78 |
| **3 APPLICATIONS OF DUNKL ANALYSIS**……………………………… | 90 |
| 3.1 Time-fractional heat equation with Caputo fractional derivative................. | 93 |
| 3.2 Time-fractional pseudo-parabolic equation with Caputo fractional derivative………………………………………………………………………. | 112 |
| 3.3 Time-fractional heat equation with bi-ordinal Hilfer fractional derivative... | 130 |
| **CONCLUSION**……………………………………………………………….. | 142 |
| **REFERENCES**……………………………………………………………….. | 143 |

**REGULATORY REFERENCES**

References to the following standards are used in this dissertation:

GOST 7.1-84. A system of standards for information, library and publishing. Bibliographic description of the document. General requirements and rules of compilation.

GOST 7.9-95 (ISO 214-76). A system of standards for information, library and publishing. Abstract and abstract. General requirements.

GOST 7.12-93. A system of standards for information, library and publishing. Bibliographic record. Abbreviations of words in Russian. General requirements and rules.

GOST 7.32-2001. Interstate standard. A system of standards for information, library and publishing. Report on research work. Structure and rules of registration.

GOST 8.417-81. The state system of ensuring the uniformity of measurements. Units of physical quantities.

**DESIGNATIONS AND ABBREVIATIONS**

|  |  |
| --- | --- |
|  | – is the set of natural numbers, which includes 0 |
|  | – is the set of all positive integers |
|  | – is the set of all real numbers |
|  | – is the set of all positive real numbers |
|  | – is the set of all complex numbers |
|  | – denotes or |
|  | – denotes |
|  | – denotes . |

**INTRODUCTION**

**The relevance of the research topic**. In this dissertation, we investigate pseudo-differential operators generated by the Dunkl operators and consider inverse source problems for the parabolic and pseudo-parabolic equations.

Pseudo-differential operators generated by the Dunkl operators were first studied by A. Dachraoui in 2001 and several boundedness results were obtained for these operators in classical Schwarz spaces and Sobolev-type spaces. This is followed by several papers where the and -boundedness of these operators were studied, but the boundedness of amplitude, transpose and adjoint operators in Schwarz spaces were not studied.

In this dissertation, we prove the boundedness of pseudo-differential, amplitude, transpose and adjoint operators with classical symbols, generated by the Dunkl operators, in Schwarz spaces. Inverse source problems for parabolic and pseudo-parabolic equations with fractional derivatives of Caputo and Hilfer in time, which had not been considered before, were also studied. It was shown that these problems are correctly solvable in the Hadamard sense and explicit form of solutions to these problems were obtained.

**The aim and objectives of the study** is to develop the theory of pseudo-differential operators generated by the Dunkl operators and to study inverse source problems for the parabolic and pseudo-parabolic equations.

**The main provisions for the defense of the dissertation:**

1. Prove that the pseudo-differential operators with classical symbols, generated by the Dunkl operators are linear continuous operators defined on Schwarz spaces.

2. Prove that the amplitude operators with classical symbols, generated by the Dunkl operators are linear continuous operators defined on Schwarz spaces.

3. Prove that the transpose operators with classical symbols, generated by the Dunkl operators are linear continuous operators defined on Schwarz spaces.

4. Prove that the adjoint operators with classical symbols, generated by the Dunkl operators are linear continuous operators defined on Schwarz spaces.

5. Define the kernels of the pseudo-differential operators generated by the Dunkl operators and show that the kernels are smooth functions.

6. Show the correct solvability of the inverse source problem for the heat equation with the Caputo fractional derivative in time, generated by the Dunkl operator.

7. Show the correct solvability of the inverse source problem for the pseudo-parabolic equation with the Caputo fractional derivative in time, generated by the Dunkl operator.

8. Show the correct solvability of the inverse source problem for the heat equation with the Hilfer fractional derivative in time, generated by the Dunkl operator.

**The object of the study** is the Dunkl operators and the pseudo-differential operators generated by the Dunkl operators.

**The subjects of the study** are to obtain boundedness results for the pseudo-differential, amplitude, transpose and adjoint operators with classical symbols, generated by the Dunkl operators, in Schwarz spaces.

**The methods of scientific research**. The dissertation uses the methods of the theory of pseudo-differential operators, the theory of partial differential equations, the theory of functions and the theory of special functions.

**Scientific novelty of the work**. The problems that considered in this dissertation are new.

The thesis deals with the Dunkl analysis. The Dunkl analysis starts from C.F. Dunkl’s works [1-5]. The framework for a theory of special functions and integral transforms in several variables related with reflection groups were built up by him. Let be a finite Coxeter group acting on a Euclidean space as a real reflection group. In his paper [2] Dunkl defines a commuting set of first order differential-difference operators (Dunkl operators) related to such a , involving a parameter . Dunkl introduces an operator (Dunkl transform) on in terms of the eigenfunctions of the Dunkl operators, in [4, р. 1213-1226]. This transformation is just the Fourier transform, if all parameters in the operators are zero, so there is a Plancherel theorem. Dunkl proves that the Plancherel theorem still holds, if all parameters are real and non-negative. The inversion theorem and the Plancherel theorem were extended by M.F.E. de Jeu, in his work [6].

We start our thesis with an outline of the general concepts (Chapter 2): the Dunkl operator, the Dunkl kernel, the Dunkl transform, and the Dunkl convolution. Moreover, some fundamental definitions from function analysis and fractional calculus are also presented here.

Chapter 3 of our thesis is dedicated to the pseudo-differential operators generated by the Dunkl operators on the real line, and we work with Dunkl analysis on the real line. This analysis was firstly introduced by A. Dachraoui in 2001 in [7]. In [7, р. 161-177] author, after carefully revising the Harmonic analysis associated with the Dunkl operators, defined two class of symbols and , , (definitions are given below) with and proved that pseudo-differential operator is continuous operator from into itself for , where is usual Schwartz space. Two class of symbols and and the operator are defined via following definitions [7, р. 161-177].

Definition 1.1. Let . The function is called a symbol in the class , if it satisfies

1. For fixed in , the function is smooth function on .

2. For fixed in , the function is smooth function on .

3. For all , there exists , such that.

for all and .

Definition 1.2. Let . The function is called a symbol in the class , if it satisfies

1. For fixed in , the function is smooth function on .

2. For fixed in , the function is smooth function on .

3. For all , there exists , such that.

for all and .

Definition 1.3. Let and . The pseudo-differential operator associated with the symbol is defined on by

where is the Dunkl kernel defined by

is the normalized Bessel function of first kind, is the Dunkl transform given by

and

is Gamma function.

Also in [7, р. 161-177], author introduced Sobolev type spaces , , , (next definition) and proved that the operator with symbol , is continuous from into , and from into , .

Definition 1.4. The space , , , is defined as the closure of a space of -functions on R with compact support, with respect to the norms

, if

and

, if

where

We proved that the pseudo-differential operator is a continuous linear operator on for and , where is a classical symbol class defined by the following definition.

Definition 1.5 (Symbol classes ).Let and . If is in and

for all and all . Then we will say that .

After, same continuity results follow for amplitude, adjoint and transpose operators defined on . Definitions of these operators can be found in Section 3.1. Additionally, we obtained some boundedness results of pseudo-differential operators generated by the Dunkl operators under certain assumptions listed below.

Definition 1.6. Let us define the space , as following

with norm

where is a space of (Lebesgue) measurable functions on the set of real numbers with the norm .

Assumption 1.7. We assume the symbol is defined as:

where is a complex valued measurable function on , such that

for all and is a continuous function.

Theorem 1.8. Let . Then the pseudo-differential operator is a bounded linear operator under Assumption 1.7 on , i.e.

Corollary 1.9. Let and are pseudo-differential operators with symbols and , respectively. Then under Assumption 1.7 their composition is a pseudo differential operator , which is continuous linear map on .

Corollary 1.10. Let . Then under Assumption 1.7 the composition of pseudo-differential operator and is a bounded linear operator on , i.e.

Assumption 1.11. We assume the symbol is defined by

satisfies

where is a continuous function.

Theorem 1.12. Let . Then the pseudo-differential operator with symbol , which satisfies Assumption 1.11, has a representation

and satisfies following inequality

Assumption 1.13. We assume the symbol is defined by

satisfies

where is a continuous function and is a constant. So, we have

Theorem 1.14. Let . Then the composition of the pseudo-differential operators and with symbols and , which satisfy Assumption 1.13, has a representation

and satisfies following inequality

In classical harmonic analysis for many different classes of symbols were studied boundedness properties of the pseudo-differential operators. As well, many results were obtained and extended to the pseudo-differential operators associated with the Dunkl operator. In [8], and -boundedness of the pseudo-differential operator associated with the Dunkl operator was studied by the authors C. Abdelkeffi, B. Amri, and M. Sifi for class of symbols , or simply , which contains symbols with property

for all and . Also, in [8, р. 1035-1050] was obtained a singular integral representation of the operator , proved that the kernel of the operator satisfies the condition of the singular integral theorem and defined kernel of the adjoint operator to the operator . The main results of the work [8, р. 1035-1050] are as follows.

Proposition 1.15. Assume that . Then there exists a continuous function on such that

for all , and we have

for all , such that the complement of is nonempty and , where

given on . Here is a constant which depends only on and measures are defined in Theorem 2.54.

Proposition 1.16. Let and be the adjoint operator of . Then we obtain

where , for all , such that the complement of is nonempty and , .

Proposition 1.17. Suppose that . Then can be extended to a bounded operator on .

Theorem 1.18. Let . Then can be extended to a bounded operator on , where .

Another work in this direction is [9] by B. Amri, S. Mustapha, and M. Sifi. In [9, р. 89-106], authors have extended -theorem of Calderón-Vaillancourt to the pseudo-differential operator associated with the Dunkl operator, on the real line.

Theorem 1.19 (Calderón-Vaillancourt). Assume that and , which is and satisfies

for all and all . Then can be extended to a bounded operator on .

Also, in [9, р. 89-106] was obtained -boundedness of the operator with symbols in , . We say that a symbol belongs to the class , if and satisfies

for all and all .

Theorem 1.20. Let , . Then can be extended to a bounded operator from into itself, for all .

In thesis, we obtained following kernel theorems:

Theorem 1.21 (Kernel of a pseudo-differential operator).Let . Then is on , and

for all and .

Theorem 1.22 (Convolution kernel of a pseudo-differential operator).Assume that . Then convolution kernel

of the pseudo-differential operator satisfies

for .

In [8, р. 1035-1050], authors have obtained this Theorem (Proposition 1.14) for the class of symbols , with , , , in which case we have

for . So, in Theorem 1.21 we are simplifying the condition

and generalizing Proposition 1.14.

Chapter 4 of our thesis is dedicated to several types of inverse source problems generated by the Dunkl operator on the real line. The Cauchy problem for the heat equation associated with the Dunkl operator

was considered by M. Rösler [10] (originally work was done in ) on a domain with initial data , where partial derivatives and the usual exponential kernel are replaced by Dunkl operators and the generalized exponential kernel of the Dunkl transform. Here the Dunkl Laplacian is defined by

for every . Then nonhomogeneous problem

was considered by H. Mejjaoli [11, 12] (also originally work was done in ) on a domain when g belongs to homogeneous and nonhomogeneous Dunkl–Besov spaces.

Here we considered inverse source problems for heat, and pseudo-parabolic equations with Caputo fractional derivatives, and heat equation with the bi-ordinal Hilfer fractional derivative, generated by the Dunkl operator.

In Section 4.1, we study direct and inverse source problems for heat equation generated by the Dunkl operator. First, we consider the Cauchy problem

(1.1)

where , , are given positive numbers, is suitable given function and is the left-sided Caputo fractional derivative (Definition 1.3.3).

A generalized solution of the Cauchy problem (1.1) is the function

satisfying the above equation.

Theorem 1.23.Let , . Then there exists a unique generalized solution of Cauchy problem (1.1). Moreover, it is given by the expression

where is the Dunkl kernel and and are Mittag-Leffler functions.

Then we studied following inverse source problem

(1.2)

where and are given suitable functions. Our aim is to find pair of functions .

A generalized solution of Inverse source problem (1.2) is a pair of functions , where

and .

Theorem 1.24. Let . Then a generalized solution of Inverse source problem (1.2) exists and is unique. And, moreover, it can be written by the expressions

and

where is the classical Mittag-Leffler function.

In Section 4.2, we study direct and inverse source problems for pseudo-parabolic equation generated by the Dunkl operator, as generalizations of previous problems, given in Section 4.1. First, we considered the Cauchy problem for the time-fractional pseudo-parabolic equation

(1.3)

where , , and is a given suitable function.

The following theorem shows that the Cauchy problem (1.3) has a unique generalized solution in the space .

Theorem 1.25. a) Let . Assume that and . Then a generalized solution of the Cauchy problem (1.3) exists, is unique, and given by the expression

(1.4)

where and are the Mittag-Leffler functions.

b) Let . Assume that and . Then the Cauchy problem (1.3) has a unique generalized solution, which is given by the expression (1.4).

Then we studied inverse source problem for the time-fractional pseudo-parabolic equation is

(1.5)

where , and are given suitable functions, and

We assume that . Then generalized solution of Inverse source problem (1.5) is the pair of functions , where and .

Theorem 1.25. We assume that . Then a generalized solution of Inverse source problem (1.5) exists, is unique, and can be written by the expressions

and

We complete our thesis with inverse source problem for heat equation with the bi-ordinal Hilfer fractional derivative generated by the Dunkl operator (Section 4.3). First, we consider direct problem.

Definition 1.27. We will call the function u a regular solution if it satisfies regularity conditions

, and ,

and the equation (1.6) for all , where and is the bi-ordinal Hilfer fractional derivative (Definition 1.3.6).

Let , , and . Our aim is to find a regular solution of the problem

(1.6)

where the functions and are given functions, and is a left-hand sided Riemann-Liouville fractional integrals (Definition 1.3.1). The following theorem demonstrates the unique solvability of the direct problem.

Theorem 1.26. We assume that and is finite for every fixed , , and . Then Problem (1.6) has a unique solution and . Moreover, it has a expression

where .

Then we consider inverse source problem. Let , , and . Our aim is to find a solution pair of the inverse source problem

(1.7)

where the functions , and are given functions. For this problem we have the following result.

Theorem 1.27. Let . We assume that and

is a finite well-defined nonzero number for every and , and . Then Problem (1.7) has a solution pair , where u is a regular solution, which are and with , and expressed by

and

**Theoretical and practical significance of the results**. The research on the topic is mainly theoretical and fundamental. Their scientific significance is due precisely to the deep level of fundamentality of the results obtained.

**The connection of the dissertation work with other scientific research works**. The dissertation work was carried out within the framework of the scientific project grant funding of research of young scientists under the project "Zhas Galym" of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan "Inverse source problems for pseudo-parabolic equations associated with Dunkl and Jacobi operators " (2022-2024, AP14972634).

**Approbation of the work**. The results of the work were presented and discussed at the following conferences:

– annual international traditional April scientific conference dedicated to the Day of Science Workers of the Republic of Kazakhstan, Institute of Mathematics and Mathematical Modeling (Almaty, 2019 – April 3-5);

– the annual international traditional April scientific conference dedicated to the Day of Science Workers of the Republic of Kazakhstan, Institute of Mathematics and Mathematical Modeling (Almaty, 2020 – April 1-3);

– 6th International Conference on Analysis and Applied Mathematics (ICAAM 2022)” (Antalya, 2022 – October 31 to November 6);

– Several times in the weekly Analysis & PDE seminar under supervision of Professor Ruzhansky (Ghent, 2021).

**Publications**. 7 works have been published (2 in journals indexed by Scopus [13, 14], and 4 in a journal recommended by the Committee for Quality Assurance in the Field of Science and Higher Education MSHE RK [15-18]).

**The structure and scope of the thesis**. The dissertation work consists of a title page, a table of contents, an introduction, three sections, a conclusion, and a list of references. The total volume of the dissertation is 148 pages with 87 references to literature.

**The main content of the dissertation:**

The introduction contains the relevance of the research topic, goals and objectives, the main provisions for the defense of the dissertation, the object and subject of the research, research methods, novelty and theoretical and practical significance of the research, the connection of the PhD thesis with other research papers, the approbation of the work, the author's publications, the scope and structure of the dissertation and content.

**In the first chapter**, we collect some basic results in the Dunkl analysis and fractional analysis. We define the Dunkl operator and the Dunkl Laplacian on the suitable spaces and consider properties of the Dunkl operator. We define the Dunkl kernel as a unique solution of the initial value problem generated by the Dunkl operator. Then we obtain series and Poisson integral representations of the Dunkl kernel. We prove that the Dunkl kernel does not have zeros for all . Then we define Dunkl and inverse Dunkl transforms on and study their properties. After we define the Dunkl transform on tempered distributions and prove that it is a continuous linear transformation on . Also, we give Taylor series generated by the Dunkl operator, as a part of Dunkl analysis.

**In the second chapter**, we consider pseudo-differential operators generated by the Dunkl operator. Some boundedness results for these operators were already known in the literature. We define amplitude, adjoint and transpose operators and prove that pseudo-differential, amplitude, adjoint and transpose operators are linear transformations on the Schwartz spaces. We also define pseudo-differential operators on tempered distributions and prove that it is a continuous linear transformation on . Then we study properties of the distributional and convolution kernels of the pseudo-differential operators. In particular, we prove Schur’s lemma. We obtain some boundedness results on spaces for the pseudo-differential operators and composition of the pseudo-differential operators, under certain assumptions.

**In the last chapter**, we study inverse source problems for Dunkl-heat and Dunkl-pseudo-parabolic equations with Caputo and bi-ordinal Hilfer fractional differential operators. For this inverse source problems, we prove well-posedness results in the sense of Hadamard. First, we consider direct problems and establish the unique existence of a generalized solution. Then we consider inverse source problems and define pair of solutions in suitable spaces. We use classical Fourier method. After we establish stability results, which means that the solution of the inverse source problems continuously depends on given data. Additionally, we consider some examples to give an illustration of our analysis.

**1 PRELIMINARY RESULTS**

In this chapter, we introduce some of the key concepts and techniques of function analysis, fractional calculus, and rational Dunkl theory, which are used in further chapters. These topics include linear operators, function spaces, and convergence in function spaces. We also discuss some of the fundamental theorems in function analysis, such as Lebesgue’s dominated convergence theorem and two types of Schwartz kernel theorem. We briefly provide the necessary definitions of fractional differential operators, such as Riemann-Liouville, Caputo, and bi-ordinal Hilfer fractional operators. Additionally, we cover major topics in rational Dunkl theory: the Dunkl operator, the Dunkl kernel, the Dunkl transform, the Dunkl convolution, and generalized Taylor formula. Some of results in Dunkl analysis is proven by ourselves, so they might be new.

**1.1 Background from elementary function analysis**

Let and be vector spaces over the same scalar field . A mapping which assigns to each element of a set a unique element is called an operator. The set on which acts is called the domain of .

Definition 1.1.1 Let and be vector spaces over the same scalar field . A function is said to be a linear operator (or a linear mapping) if is a subspace of and

for every and every vectors .

We often write , rather than . A linear mapping is called a linear functional and is called a function (real or complex valued).

Definition 1.1.2. A nonnegative function on a vector space is called a seminorm if it satisfies the following conditions:

1. for all and all .

2. for all .

Definition 1.1.3.A seminorm on a vector space is called a norm if

Definition 1.1.4 (Space )For an open set , the space of smooth compactly supported functions is defined as the space of smooth functions with compact support. Here the support of is defined as the closure of the set where is non-zero, i.e., by

Example 1.1.5. The function

with supp belongs to .

Definition 1.1.6 (Convergence in ).We say that in if , if there is a compact set such that supp for all , and if

for all .

Definition 1.1.7 (Distributions **).** The space is the space of continuous linear functionals on . This means that if it is a functional such that:

1. is linear, i.e., for all and all .

2. is continuous, i.e., in whenever in .

Definition 1.1.8 (Schwartz space **).** The Schwartz space is the topological vector space of functions such that and

for all . The seminorms on the space are defined by

(1.1.1)

for all and .

We say that the function belongs to the Schwartz space if and . The requirement

(1.1.2)

can be replaced by the condition

(1.1.3)

for all and . Let inequality (1.1.3) hold for all and . Then we obtain

which leads (1.1.2). On the other hand, if (1.1.2|) holds, then Newton’s Binomial Theorem gives (1.1.3).

Example 1.1.9. The function belongs to . More generally, if is any polynomial, then belongs to .

Example 1.1.10. The function

does not belongs to for any since does not decay to zero as .

Definition 1.1.11 (Convergence in ) We will say that in as , if , and if for all , where seminorms are defined by the formula (1.1.1).

Definition 1.1.12. A linear operator

is continuous if in implies in .

Let us give here very useful theorem, formulated as following:

Theorem 1.1.13 (Lebesgue’s dominated convergence theorem) [19] Let be a sequence of measurable functions on such that pointwise almost everywhere on as . Suppose there is an integrable function such that for all . Then is integrable and

Proposition 1.1.14. [19, p. 240] The space is sequentially dense in , i.e., for every there exists a sequence such that in as .

Definition 1.1.15 (Tempered distributions ). We define the space of tempered distributions as the space of all continuous linear functionals on . This means that if it is a functional such that:

1. is linear, i.e., for all and all .

2. is continuous, i.e., in whenever in .

We can also define the convergence in the space of tempered distributions. Let . We will say that in as if in as , for all . Functions in are called the test functions for tempered distributions in . Another notation for will be .

Example 1.1.16. The function defined by the Riemann integral

is a tempered distribution.

Proposition 1.1.17 (Continuous inclusion ). [19, p. 242] The inclusion is continuous, i.e., in implies in .

The following Theorems proved in H. Gask’s work [20] published in 1960.

Theorem 1.1.18 (Schwartz kernel theorem I) For any separately continuous bilinear functional on there exists precisely one distribution in such that

for all in .

Theorem 1.1.19 (Schwartz kernel theorem II). For any separately continuous bilinear functional on there exists precisely one distribution in such that for all in .

Now, let us provide some necessary information about Gamma function and Bessel functions.

**1.2 The Dunkl analysis**

Let us first give necessary information about Gamma function. The Gamma function is a special function, which is defined for all complex numbers except the non-positive integers. For complex numbers such that the gamma function is given by

We have for the Gamma function the following recurrence relation:

(1.2.1)

We can show (1.2.1) by Integration by parts:

From (1.2.1) we obtain an interesting formula

The Gamma function is the generalization of the factorial to complex numbers. Indeed,

Example 1.2.1 and .

Definition 1.2.2 ([21]) The Dunkl operator is the differential-difference operator

defined by

(1.2.2)

for every and the Dunkl Laplacian

is defined by

for every . Here we understand as a composition of the operators and , i.e., .

Remark 1.2.3 From Mean Value Theorem, we readily see that

for some . This gives

(1.2.3)

for some and

Also, the expression (1.2.3) gives a following result

(1.2.4)

for .

Remark 1.2.4 In general, the Dunkl operator is defined for every , but in this thesis we are interested only real .

The Dunkl operator is not only well defined from to , we can consider as a domain of more important spaces, as with , , and .

Lemma 1.2.5 ([21, p. 6]) If with , then .

Lemma 1.2.6 ([22]) The Dunkl operators map the following function spaces into themselves:

Proposition 1.2.7 [23] Let . For , the following differential equation with initial condition:

(1.2.5)

has a unique solution given by

(1.2.6)

where is called the normalized Bessel function of first kind.

Proof. Let be a solution of the problem (1.2.5). We can write

where

Then after putting to the equation (1.2.5) we obtain an equivalent equation

So, satisfies Bessel type equation

(1.2.7)

The Bessel type equation (1.2.7) has the unique solution ([24])

for all . The function is called the normalized Bessel function of first kind and has the following series representation

The function has the following properties ([24, p. 9]):

1. For all , the function is an even -function on .

2. For all , the function is an even entire function on .

Derivative of the function gives us

Thus, we have

from which we obtain (1.2.6). The proof is complete.

So, the function has the following properties:

1. For all , the function is a -function on .

2. For all , the function is an entire function on .

3. .

4. .

where .

Remark 1.2.8 In the case , the equation (1.2.5) turns into an ordinary differential equation

which has a solution

Remark 1.2.9 The function is called the Dunkl kernel in the literature.

Corollary 1.2.10 [25] Let . Then the Dunkl kernel has the following series representation

(1.2.8)

where is a generalized factorial, defined by

(1.2.9)

Proof. The proposition follows from next short calculations

The proof is complete.

Remark 1.2.11 [25, p. 372] The generalized factorial also has a recurrent formula

(1.2.10)

where is for even and for odd .

Lemma 1.2.12 Suppose that and . Then

(1.2.11)

Proof. The Lemma can be proved by using mathematical induction. The Basis for induction is clearly true, since

For induction step, suppose is true. Then inequality (1.2.11) holds for , which is clear from

So, the induction step holds. The proof is complete.

Corollary 1.2.13 Let . Then Corollary 1.2.10 leads that

Proof. A short calculation. Here we have used property of the Gamma function and (1.2.8). The proof is complete.

Lemma 1.2.14 Assume that and . Then

for all and . In particular

when .

Proof. Using expression (1.2.8) of the Dunkl kernel and Lemma 1.2.12 we obtain

Differentiation times from gives us

Taking into account this fact we reach the inequality

The proof is complete.

Proposition 1.2.15 Let . Then for every , the Dunkl kernel has the Poisson integral representation

(1.2.12)

for all .

Proof. Let . Then for every , the function has the Poisson integral representation

for all ([24, p. 11]). Thus,

The proof is complete.

From integral representation of the Dunkl kernel (1.2.12) it is convenient to obtain another property of the Dunkl kernel

(1.2.13)

Corollary 1.2.16 Let and . Then we have the following estimates for the Dunkl kernel

for all and . In particular, we have

(1.2.14)

for all , when .

Proof. A short calculation. Here we have used integral representation of the Dunkl kernel (1.2.12) and the fact

The proof is complete.

Now, we are able to give a short representation for Dunkl kernel

where is the Dunkl’s intertwining operator, defined by

on the space of smooth functions, i.e., .

Lemma 1.2.17 Let . The function does not have zeros.

Proof. (1) Let first , then the Dunkl kernel is exponential function

So, we have to consider two cases and . Using definition of the Bessel function of the first kind we can rewrite expression (1.2.6) as

where is the Bessel function of the first kind with , given by

Here we need not worry about , because and we are going to consider only positive , it is obvious form equation .

(2) Now, let us consider case when . Then we use the theorem ([26]): Between two consecutive positive zeros of the Bessel function , there is one and only one root of the Bessel function , and vice versa. Thus, when , the Bessel functions and have many zeros, but they can not be equal to zero at the same time. So, in not equal to zero for all , when .

(3) Finally, in a case , we prove lemma using recurrent formula ([26, p. 20])

as following. Let us fix arbitrary , and for some we have

If we continue this process until , we obtain for some and it is contradiction to the theorem given in (2). The proof is complete.

Let , be the space of measurable functions on such that

and

where

(1.2.15)

Remark 1.2.18 Note that, in the case , expression (1.2.15) gives us

thus is the usual space.

Lemma 1.2.19 (Hölder’s inequality). We assume that , and . Then we have

(1.2.16)

Proof. Let and . Then

where we have used classical Hölder’s inequality. The proof is complete.

Lemma 1.2.20 We have with continuous embedding, i.e., in implies that in for all .

Proof. Let . Then we obtain

where and . Let in . We assume that . Then we are able to calculate

Now, let . Then

since in . The proof is complete.

The Dunkl kernel leads to the Dunkl transform , which is defined by the formula

(1.2.17)

for and using (1.2.14) we obtain

for any . Thus

Remark 1.2.21 For , the Dunkl transform is the Fourier transform

The inverse Dunkl transform is defined by

Theorem 1.2.22

1) [6, p. 159] The Dunkl transform is a homeomorphism of ;

2) (Plancherel theorem) [6, p. 160] The Dunkl transform has a unique extension to an isometric isomorphism of , i.e.

for all ;

3) (Inverse Dunkl transform) [6, p. 159] For all with ,

The space is the space of continuous functions on which vanish at infinity.

Lemma 1.2.23 (Riemann-Lebesgue lemma) [6, p. 156] The Dunkl transform maps into .

Proposition 1.2.24 (Product Rule for the Dunkl operator). If is even, then . If and at least one of them even, then

(1.2.18)

Proof. The first statement of the proposition is obvious, because it follows from the definition of the Dunkl operator (1.2.2). Without losing generality, we may assume that is even function, then the second statement of the proposition follows from the following equations

The proof is complete.

Natural question, after Proposition 1.2.24, is what if one of the functions is odd? Answer to this question is

if is odd function. Now, we are ready to define Product Rule for the Dunkl operator for any relevant function. As we can write any function in a form

we obtain

Thus, Product Rule for the Dunkl operator is

for all .

Proposition 1.2.25 Let . Then for every and we have

(1.2.19)

or

where and .

Proof. Using definition of the Dunkl operator we obtain

Then we calculate

while the other parts yields

We can finish our proof taking into account that our last integral equals to zero, since under integral we have an odd function. The proof is complete.

Corollary 1.2.26 Let . Then for every and we obtain

(1.2.20)

for any .

Let . Then the Dunkl transform has the following properties ([23, p. 109]):

and .

2. ;.

3. The Dunkl transform leaves invariant.

Proof. Let . Then

for all , thus

Using Proposition 1.2.7 and Proposition 1.2.25 we obtain

and

To prove the last property, notice that it suffices to prove that is bounded for arbitrary . Applying previous property times and Corollary 1.2.26 we have

Consequently, we obtain

and

(1.2.21)

for all , where

(1.2.22)

Here we have used (1.2.4). The proof is complete.

Proposition 1.2.27 The Dunkl transform is a linear continuous map on .

Proof. As the Dunkl transform is a linear map, so we prove only its continuity. Let in as . Then using (1.2.21) and (1.2.3) we obtain

where is constant defined by (1.2.22). The proof is complete.

Lemma 1.2.28 (Multiplication formula for the Dunkl transform). Let . Then

Proof. Applying Fubini’s theorem we have

where we have used the Dunkl kernel’s property. The proof is complete.

Let . Note that the Dunkl operator is skew symmetric with respect to the -norm associated to the measure , i.e.

where .

Now, we show that the Fourier transform can be extended from to by duality [27]).

Definition 1.2.29 (Dunkl transform of tempered distributions) If , we can define its (generalised) Dunkl transform by setting

(1.2.23)

for all .

We can interpret functions in , , as tempered distributions. If , we define functional by

for all . By Hölder’s inequality, we define that

for . Hence, is well defined in view of the simple inclusion , for all (see Lemma 1.2.20). Therefore,

Then using Lemma 1.2.28, we obtain

Hence, we have

Proposition 1.2.30 (Dunkl transform on) The Dunkl transform is a continuous linear operator from to .

Proof. Since the Dunkl transform leaves invariant the Schwartz space , (1.2.23) is well defined. Let , and . Then we have

and

since in and the Dunkl transform is a linear continuous map on (Proposition 1.2.27). Thus, defined by (1.2.23). Now, it follows that it is also continuous as a mapping from to , i.e., if in , then

which means that in . The proof is complete.

The following properties holds (see [26, p. 10]):

1. is a topological isomorphism of onto itself.

2. , for all .

3. , for all .

We also define the inverse Dunkl transform on using (1.2.23), i.e.

Theorem 1.2.31 (Fourier inversion formula for tempered distributions). Operators and are inverse to each other on , i.e.,

Proof. Let and . Then we have

Same calculation can be used for second equation. The proof is complete.

We have the following product formula for the function with and parameter ([24, p. 13]):

for , where

(1.2.24)

Here is the indicator function of and

denotes the area of the triangle with sides . The function satisfies the following properties ([24, p. 13-14]):

1. For all ,

2. We have for :

3. We have for all :

For our convenience, we fix some notations. For all , we put

and

Theorem 1.2.32 [28]

1) Let and . Then the Dunkl kernel satisfies the following product formula:

(1.2.25)

for , where

with kernel

where is the Bessel kernel (1.2.24).

2) The measures have the following properties:

1. supp for ,

2. for all .

Remark 1.2.33 In Theorem 2.52, is the Dirac measure. So, we have the following statements:

1. If , then

2. If , then

Remark 1.2.34 Let . Then from

we obtain

(1.2.26)

Lemma 1.2.35 Let . Then

Furthermore, we have

Proof. For any a short calculation gives us the following equalities

and

Then using property of the function we obtain

Thus, we have

The proof is complete.

For all and a continuous function on , we define

(1.2.27)

The operators are called Dunkl translation operators on .

Proposition 1.2.36 [29] The operators have the following properties:

1. For all and , , we have

2. For all and , we obtain

For two continuous functions and on with compact supports, we define a convolution product by

where is the Dunkl translation operator on .

Remark 1.2.37 Note that is the standard convolution .

Proposition 1.2.38 [29, p. 21] (i) Let and satisfy Then the map can be extended to a continuous map from to , and

(ii) For any and , we have

A polynomial is a function of the form

where are constants. Polynomials are convenient to work with, because their values can be calculated easily. Therefore, they have been extensively used to approximate more complicated functions. Taylor’s theorem is one of the oldest and most important results on this question.

From basic theory of calculus, it is known that, if we consider the function given by the sum of a power series with radius of convergence ( may be )

then the power series has the form

(1.2.28)

where all derivatives should exist.

Now, question is what if we use the Dunkl operator instead of usual derivative, as natural generalization of usual first order derivative. In which case, we obtain more general form than (1.2.28), as stated in Lemma 1.2.38.

Lemma 1.2.39 Let . Suppose that is an analytic function

on with radius of convergence [ may be ]. Then the original power series has the form

(1.2.29)

Proof. Taking into account the fact

(1.2.30)

which we will prove later, and applying the Dunkl operator to the function times we obtain

where the polynomial has a property . Thus , which proves the (1.2.29). Now let us prove the (1.2.30). In (1.2.30), the second one is not obvious, so we will focus on that one. To prove we use mathematical induction. Let and , then and , respectively. After we suppose is true. Then we have

Finally, using recurrent formula for ([25, p. 372]) we obtain , which completes our proof. The proof is complete.

Now, we assume that and is defined on and the derivative exists on . Then we can define -order Taylor polynomial for about , i.e.

and for , remainder is defined by

(1.2.31)

So, from (1.2.31) we obtain

The next question is, does such a remainder exist? Classical Taylor’s Theorem states that such remainder exist, for , i.e.

Theorem 1.2.40 We assume that is defined on , where , and the derivative exists on . Then for each in there is some between and such that

Corollary 1.2.41 We suppose that is defined on , where , and the derivative exists on and are bounded by a single constant . Then

for all .

So, can we extend this results in a case . This extension was obtained by M.A. Mourou in [30] in 2003. He extended Theorem 1.2.40 and Corollary 1.2.41 to a first-order general differential-difference operator

on the real line which in the particular case, when , , gives the Dunkl operator . He established a generalized Taylor formula with integral remainder. Before, let us introduce some notations from [30, р. 343-353].

Let be the subset of defined by . Then using recursive integral formulae

and

we define sequences of functions , , on . The central result of the paper [30, р. 343-353] is stated as following.

Theorem 1.2.42 Let . Then we can obtain the generalized Taylor formula for the Dunkl operator with integral remainder:

for all , where

Corollary 1.2.43 Let . Assume that there are such that

for all . Then there exists an such that

for . Moreover, the series converges uniformly for .

**1.3 Fractional differential operators**

In this section we give the definitions of the Riemann-Liouville fractional integrals and fractional derivatives, Caputo and bi-ordinal Hilfer fractional derivatives on a finite interval of the real line and present some of their properties. Also, we here present some necessary information about Mittag-Leffler functions.

Definition 1.3.1 [31] Let be a finite interval on the real axis and . The left-hand sided and the right-hand sided Riemann-Liouville fractional integrals of order are defined by

and

respectively. Here is Euler’s gamma function.

Definition 1.3.2 [31, p. 70] The left-hand sided and the right-hand sided Riemann-Liouville fractional derivatives of order , which are expressed by

and

respectively. Here .

The Riemann-Liouville fractional integrals and derivatives have the following properties:

1. If and , then ([31, p. 71])

and

2. Let , , , and , . Then we have [31, p. 73]

3. Let and . If and , then the equalities

and

hold almost everywhere on ([31, p. 74-75]).

Definition 1.3.3 [31, p. 91] The left-hand sided and the right-hand sided Caputo fractional derivatives of order (), which are defined by the formulas

and

respectively. Here

Definition 1.3.4 [32] Let be a Banach space. We say that if and .

Remark 1.3.5 [31, p. 92] Assume that and . Then Caputo fractional derivative exists almost everywhere on . Moreover, the left-sided and right-sided Caputo fractional derivatives are represented by

and

respectively. Here .

The bi-ordinal Hilfer fractional derivative

where , and , is introduced by R. Hilfer in [33] in 2000, as a new generalization of the Riemann-Liouville derivative. A few applications of this operator were investigated by numerous mathematicians, for example [34, 35]. A generalization of this operator

where , , and , is introduced in Toshtemirov’s PhD dissertation [36]. In the PhD dissertation in chapter IV, the author considered direct and inverse problems for the pseudo-parabolic equation with the bi-ordinal Hilfer fractional derivative.

Definition 1.3.6 The left-hand sided and right-hand sided bi-ordinal Hilfer fractional derivatives ([36, p. 16]) of orders () and () type are expressed by

and

respectively.

Remark 1.3.7 In the case , we obtain the classical Riemann-Liouville fractional derivatives and when , they turn into the Caputo fractional derivatives (Remark 1.3.5).

Remark 1.3.8 The left-sided bi-ordinal Hilfer fractional derivative can be rewritten as following

for , where and , which have the properties , and , .

Now, we present the definitions and some properties of classical Mittag-Leffler functions. More detailed information may be found in the book [31, p. 40-42].

The Mittag-Leffler function is a special function, a complex function which depends on . It may be defined by the following series when the real part of is strictly positive

where is the gamma function. When , we obtain special case of the Mittag-Leffler function defined by

Here we able to see that . In particular case, when , we have

For (not true for ) we have the following estimates ([37]) for Mittag-Leffler function

(1.3.1)

holds over , with optimal constants. Then it follows that

(1.3.2)

The Riemann-Liouville fractional integral of the Mittag-Leffler function with special parameters also yields a function of the same kind ([31, p. 78])

Theorem 1.3.9 [38] Suppose that , and, . Then there exists a positive constant such that

for all and .

**2 PSEUDO-DIFFERENTIAL OPERATORS ASSOCIATED WITH THE DUNKL OPERATOR**

Let us have a differential operator

We can describe as a pseudo-differential operator using classical inverse Fourier transform as following

So, we have

(2.1)

for all and the function is called the symbol of the operator . Then the idea of pseudo-differential operators is to consider operators of the form (2.1) where is a more general sort of function. Thus, pseudo-differential operator is defined by

for all , where the symbol from following class:

Definition 2.1 (Symbol classes ) Let and . If is in and for all and all . Then we will say that .

Now, we would like to give a motivation to study pseudo-differential operators associated with the Dunkl operator. If we consider the Dunkl operator

with , then we are not able to calculate its symbol using classical inverse Fourier transform , it is clear from

for all . On the other hand, if we use the inverse Dunkl transform instead of classical inverse Fourier transform we obtain

So, the Dunkl operator is a pseudo-differential operator with symbol in the Dunkl setting.

**2.1 Pseudo-differential and Amplitude operators on Schwartz spaces**

In this section, we define amplitude, adjoint and transpose operators and prove that pseudo-differential, amplitude, adjoint and transpose operators are linear transformations on the Schwartz spaces.

Lemma 2.1.1 Let and . Then

a. for every , the function is belongs to the . Moreover, we have

b. for every , the function is belongs to the , i.e.

Proof. Let and . Then for every , the function is belongs to the . Since

and

where for every fixed and we can find positive integer , which satisfy the inequality . Moreover, we have

(2.1.1)

Now, let us have the function for every . Then we are able to calculate

and

The proof is complete.

Lemma 2.1.2 Let and in as . Then we obtain

1. in as , for every .

2. in as , for every fixed .

Proof. Let us show that for every , we obtain in as . Using previous calculations, we obtain

as for all . However, we have in as only for every fixed , i.e.

as . The proof is complete.

Lemma 2.1.3 We assume that and . Then

for all .

Proof. Let assumptions of the lemma holds. Then using Mean Value Theorem as following

for some , we are able to calculate

and

Then taking absolute value from last equation and supremum respect to the variable we obtain

Now, let us prove that

Indeed

and

since . The proof is complete.

Lemma 2.1.4 We suppose that and in as . Then

as for all .

Proof. The proof of this lemma is the same as the proof of Lemma 2.1.3. The proof is complete.

If , it is convenient to denote by the corresponding pseudo-differential operator defined by

(2.1.2)

where is the Dunkl transform (1.2.17) of , is the Dunkl kernel (1.2.6), and a weighted Lebesgue measure (1.2.15) on .

Theorem 2.1.5 (Pseudo-differential operators on ) Assume that and . Then .

Proof. Let and . To show absolute convergence of the integral (2.1.2), we calculate

as and fixed here, we can find suitable . Using Lebesgue’s dominated convergence theorem and properties of the Dunkl kernel we obtain

and

Thus . The same is true for all of its other derivatives, it is obvious from last equation, which implies that . Let us show now that . In fact, we have

Then for arbitrary we obtain

which implies

for all , where

(2.1.3)

There we have used Lemma 2.1.3. Then taking into account inequality (2.1.1), we can see that is bounded for all . This completes the proof.

Theorem 2.1.5 shows that the operator is a linear operator defined on the Schwartz space . Actually, can prove that is a linear continuous operator on the Schwartz space .

Proposition 2.1.6 Suppose that and . Then the pseudo-differential operator is a continuous linear operator on .

Proof. Let and in as . Then using Proposition 1.2.27 and Lemma 2.1.4 we have

for every fixed , where is constant defined by (2.1.3). The proof is complete.

Proposition 2.1.7 Assume that we have a sequence of symbols which satisfies

for all , all , and all , with constants independent from and , and such that and all of its derivatives converge to and its derivatives, respectively, pointwise as . Then for any we have

in .

Proof. From Theorem 2.1.5, for every and we have . Then using Lebesgue’s dominated convergence theorem, we obtain

since there is an integrable function such that

A short calculation gives us

Then using Lemma 2.1.4 we have

for every fixed , where is constant defined by (2.1.3). The proof is complete.

Let us introduce more general symbol class that introduced in Definition 2.1.

Definition 2.1.8 Let and . The class is the space of all functions which are and satisfy

for all and all .

By analogy of (2.1.2), for all we may also formally define corresponding operator

to the , which is called amplitude.

Theorem 2.1.9 Let and . Then the operator is a linear continuous operator on .

Proof. First, we prove that the function

is rapidly decreasing in both variable and . To see it we calculate

and

Thus,

and is rapidly decreasing if . Similarly, we have

for all , so belongs to . Using same technique with little modifications we obtain

and

Hence, Lemma 2.1.3 leads that is bounded for arbitrary and . So,

is absolutely integrable for , if and

is bounded. Thus, .

After successfully proving the first part of the theorem, we now turn our attention to the second part, which deals with continuity of the operator A on Schwartz spaces. Assume that there exists a sequence , which converges to in . Then we need to show that in as . Let’s first prove that if , as , then in fro fixed , indeed from

we have

which affirms our statement. Then from

taking absolute value, we obtain

This proves our statement.

We are now in a good position to compute adjoints of pseudo-differential operators. We say that is the adjoint ([39]) of if

for all , where

Lemma 2.1.10 Let . Then , where .

Proof. Assume that . Since

to prove this, all we need to do is reverse the order of integration. However, it’s important to note that this is not a common application of Fubini’s theorem, as the triple integral is typically not absolutely convergent. Instead, we utilize Fubini’s theorem on the double integral

that is absolutely convergent, to obtain

where

The function is a rapidly decreasing function by proof of Theorem 2.1.9, so we can apply Multiplication formula for the Dunkl transform (Lemma 1.2.28) as following

Therefore

and

so that with as claimed. The proof is complete.

Corollary 2.1.11 Let and . Then is a linear continuous map on .

We say that is the transpose ([39, p. 289]) of if

for all , where

Lemma 2.1.12 Let . Then , where .

Proof. The proof of this lemma is same as for Lemma 2.1.10, except last part. Suppose that . Then following proof of Lemma 2.1.10, we obtain

Hence,

so that with . The proof is complete.

Corollary 2.1.13 Let and . Then is a linear continuous map on .

Before, we proved that the pseudo-differential operator with symbol is a continuous linear operator on (see Proposition 2.1.6) in the sense that if in then in . Suppose there is an adjoint of the operator . Then we can extend to act on distributions as following:

Definition 2.1.14 Let . Then for all we define by the formula

where

The linear functional on defined in this way is continuous on since is continuous.

Proposition 2.1.15 Let and . If and then . Moreover, the operator is continuous.

Proof. Let . Let us prove linearity of the functional . Using definition of the tempered distributions, we have

for all and all . Now we prove continuity of the functional . Let we have in , then we have in and

These two propositions give us . Let we have in . Then to show the continuity of the operator it is enough to calculate

The proof is complete.

**2.2 Kernel of pseudo-differential operators**

Let . Assume that is an integral operator on some space of functions on that

If , we have

or in the language of distributions,

(2.2.1)

If we suppose that is a continuous linear map form to , then the Schwartz kernel theorem implies that there exists unique kernel such that holds for all and is called the distributional kernel of .

Now, it is easy compute the distributional kernel of a Kernel of pseudo-differential operator. Indeed, if , then

from which it follows that the kernel of is

with the appropriate interpretation as a distributional integral. On the other hand, if we regularize the integral

(2.2.2)

for , we able to consider the kernel as a function. Let be a positive even integer, then we have

Then inserting this expression into (2.2.2), replacing , and integrating by parts, we obtain

where . Then this integral is absolutely convergent, if we set , and

(2.2.3)

is a convergent integral for .

Theorem 2.2.1 Let . Then , given by (2.2.3), is on , and

for all and .

To prove Theorem 2.2.1, we shall use the following simple assertion.

Lemma 2.2.2 Let . Then have

and

for all and all .

Proof. Let . Then from (1.2.3) we can readily see that

for some . Thus,

for all and all . Second inequality can be proved using same method. The proof is complete.

Now, let us give the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. From the kernel expression (2.2.3), for all we have

and

for large enough . So, . Applying Theorem 1.2.32 for we obtain

Then

since from Theorem 1.2.32 it is known that supp for , after taking into account that is the even function we obtain

for large enough . The proof is complete.

The singular support of a distribution is the complement of the largest open set on which is a function.

Corollary 2.2.3 (Singular supports) Let is a pseudo-differential operator with symbol . Then for every we have

Lemma 2.2.4 (Schur’s lemma) Let be a continuous function satisfying

and let be the linear operator with Schwartz kernel :

Then is a bounded linear operator on with

Proof. Let . Then using Hölder’s inequality (1.2.16) for , we obtain

Integrating with respect to variable , we have

The proof is complete.

Now, let us define convolution kernel. Thanks to the expression (2.2.3), we are able to calculate

where

The last integral exists, since .

According to previous calculations we can formally write pseudo-differential operators in various ways:

Theorem 2.2.5 (Convolution kernel of a pseudo-differential operator) Assume that . Then convolution kernel

of the pseudo-differential operator satisfies

for .

Proof. Using integral representation of the convolution kernel we obtain

where .Hence, we have

for . The proof is complete.

Proposition 2.2.6 Let . Then for any continuous linear pseudo-differential operator

with symbol there exists a unique convolution kernel such that

Proof. Let . Then rewriting the expression

of the pseudo-differential operator , we obtain

After taking into account discussion in the beginning of this section about kernel of the operator and Schwartz kernel theorem I (Theorem 1.1.18) we can complete our proof. The proof is complete.

**2.3 Boundedness of pseudo-differential operators generated by the Dunkl operator**

In this section, we introduce a space by Definition 2.3.1 and obtain some boundedness results for pseudo-differential operators and composition of the pseudo-differential operators generated by the Dunkl operator in this space. The methods used in this section are taken from [40, 41].

Definition 2.3.1 Let us define the space , as following

with norm

(2.3.1)

Assumption 2.3.2 We assume the symbol is defined as:

(2.3.2)

where is a complex valued measurable function on , such that

for all and is a continuous function.

Remark 2.3.3 The integral (2.3.2) exists, because

Theorem 2.3.4 Let . Then the pseudo-differential operator

is a bounded linear operator under Assumption 2.3.2 on , i.e.

Proof. Let . Then by the definition of the pseudo-differential operator we obtain

using above assumption and Fubini’s theorem. After applying the Dunkl transform to the both sides of the equation we have

Hence, taking integral from both sides we obtain

This completes proof of the theorem.

Let . The composition of two pseudo-differential operators

and

with the symbols and respectively, is

where

Thus,

is a pseudo-differential operator with symbol

Now, let us discuss about existence such an integral under Assumption 2.3.2. Let

(2.3.3)

and

(2.3.4)

where and are complex valued measurable functions on , such that

for all and are continuous functions. Then by using integral expressions (2.3.3) and (2.3.4) of and respectively, we obtain

After taking absolute value from both sides of the equation, as following

we can see that the is a bounded function.

Corollary 2.3.5 Let and are pseudo-differential operators with symbols and , respectively. Then under Assumption 2.3.2 their composition is a pseudo-differential operator , which is continuous linear map on .

Corollary 2.3.6 Let . Then under Assumption 2.3.2 the composition of pseudo-differential operators and is a bounded linear operator on , i.e.

Proof. Let . Then we have

and

where we have used (1.2.26). Then taking absolute value and integrating we have

The proof is complete.

Assumption 2.3.7 We assume the symbol is defined by

satisfies

where is a continuous function.

Theorem 2.3.8 Let . Then the pseudo-differential operator with symbol , which satisfies Assumption 2.3.7, has a representation

and satisfies following inequality

Proof. By using Assumption 2.3.7 we have

Thus, applying the Dunkl transform we obtain

(2.3.5)

By taking integral from both sides of the above equation, we able to calculate

and

Further, it can be written as

The proof is complete.

Corollary 2.3.9 Let

where is a continuous function. Then the operator

is a continuous linear operator from to . Moreover, we have

where .

Proof. Let . Then

And

Hence, symbol of the pseudo-differential operator expressed by the form

Then by applying Theorem 2.3.8, we obtain

where . The proof is complete.

Assumption 2.3.10 We assume the symbol is defined by

satisfies

where is a continuous function and is a constant. So, we have

Theorem 2.3.11 Let . Then the composition of the pseudo-differential operators and with symbols and , which satisfy Assumption 2.3.10, has a representation

and satisfies following inequality

Proof. Let . Then we have

Now an application of the Dunkl transform gives us

Then by using (2.3.5), we obtain

so that

Thus, we have

This completes proof of the theorem.

**3 APPLICATIONS OF DUNKL ANALYSIS**

As an application of Dunkl analysis, we consider inverse source problems for time-fractional nonhomogeneous heat and pseudo-parabolic equations with Caputo fractional derivatives , , and bi-ordinal Hilfer fractional derivatives , , , generated by the Dunkl operator (1.2.2). Section 3.1 deals with the inverse source problem for the time-fractional nonhomogeneous heat equation with Caputo fractional derivative ([13, р. 393-407]), in Section 3.2, we study inverse source problem for the time-fractional nonhomogeneous pseudo-parabolic equation with Caputo fractional derivative ([42]), and in Section 3.3, we consider the time-fractional nonhomogeneous heat equation with bi-ordinal Hilfer fractional derivative.

Remark 3.1 Author of thesis have also published several papers ([14, р. 3347-3356; 15, р. 11-15; 16, р. 92-108; 17, р. 76-81; 18, р. 50-54; 43]).

Inverse source problem firstly was studied by W. Rundell and D. L. Colton in [44]. They considered the evolution type equation

(3.1)

in a Banach space , where is linear operator in and is a constant vector in , with conditions

Using semigroups of operators Rundell proved a general theorem about the existence of a unique solution pair of the problem, which then was applied to equations of parabolic and pseudo-parabolic types. A.I. Prilepko and I.V. Tikhonov in their work [45] studied several inverse source problems for the equation (3.1) when the non-homogeneous term is represented in the form , where is known operator and the element is unknown, and is a closed linear operator from into . They applied obtained results to the transport equation. In [46] I. Bushuyev considered inverse source problems for the equation (3.1), where the unknown source depends on time, under a sufficient condition, with the linear elliptic partial differential operator of order with the bounded measurable coefficients such that

for all , . When unknown source given by the general form there is no closed theory. Known results deal with separated source terms. I.V. Tikhonov and Yu.S. Eidelman [47] considered inverse source problems for the generalization of the equation (3.1) of the form

for some positive integer and some real number with an unknown parameter and a closed linear operator in the Banach space under the Cauchy conditions and “over-determination condition” (also in the Banach space). For the Laplace operator which is one of the most interesting examples in Physics, M. Choulli and M. Yamamoto in [48] established the uniqueness and conditional stability in determining a heat source term from boundary measurements with , where is known. M. Yaman and Ö. F. Gözükızıl in [49] studied asymptotic behaviour of the solution of the inverse source problem for the pseudo-parabolic equation

with an integral over-determination condition.

Fractional derivatives and fractional partial differential equations have received great attention both in analysis and application, which are used in modeling several phenomena in different areas of science such as biology, physics, and chemistry, so the fractional computation is increasingly attracted to mathematicians in the last several decades. K. Sakamoto and M. Yamamoto in [50] considered inverse source problem for the time fractional parabolic equation

where is the Caputo derivative defined by

and is a symmetric uniformly elliptic operator. The authors proved that the inverse problem is well-posed in the Hadamard sense except for a discrete set of values of diffusion constants using final overdetermining data. M. Yaman in [51] studied blow-up solution and stability to inverse source problem for the pseudo-parabolic equation

with the integral overdetermination condition. M. Slodička in [52] considered inverse source problem for the equation (3.1), when is a linear differential operator of second-order, strongly elliptic, and the right-hand side is assumed to be separable in both variables and , i.e., (in this case is unknown). M. Slodička and K. Šišková in [53] studied inverse source problem for a semilinear time-fractional diffusion equation of second order in a bounded domain in

with a linear second order differential operator in the divergence form with space and time dependent coefficients. Authors showed the existence, uniqueness and regularity of a weak solution ([53, p. 1658]). Also, the inverse source problem for the heat equation

with the Caputo fractional derivative was considered by M. Ruzhansky, N. Tokmagambetov, and B.T. Torebek in [54] in 2019, where is a linear self-adjoint positive operator with a discrete spectrum on a separable Hilbert space . Authors obtained unique solution pair of the given equation under the conditions

One of the recent papers for inverse source problems for pseudo-parabolic equations with fractional derivatives is [55]. M. Ruzhansky, D. Serikbaev, B.T. Torebek and N. Tokmagambetov have considered solvability of an inverse source problem for the pseudo-parabolic equation with the Caputo fractional derivative of order

where be a separable Hilbert space and , be operators with the corresponding discrete spectra on . The authors obtained well-posedness results.

A number of articles address the solvability of the inverse problems for the diffusion ([56-59]), pseudo-parabolic ([60-67]) and sub-diffusion equations ([68-73]) and fractional diffusion equations ([74-76]). We also would like to note recent works [77-81], where inverse source problem was the subject of investigation.

An important motivation for studying non-local parabolic type problems for the Dunkl operators is related to their relevance for the evaluation analysis of many-body quantum systems of the Calogero-Moser-Sutherland type. These quantum systems describe algebraically integrable systems and are of considerable interest in mathematical physics, especially in con-formal field theory. For the related references we refer the reader to the book [82]. The semigroups (the solution of the heat equation associated with the Jacobi-Dunkl operator ) generate a new family of Markov processes on the real line. On some Riemannian symmetric spaces this process is the radial part of the Brownian motion for particular values of [83].

**3.1 Time-fractional heat equation with Caputo fractional derivative**

In this section we prove the existence and uniqueness of the solution of the Cauchy problem

where , , and are given positive numbers and its limit case, when ,

Then we study, the main problem of this section, the inverse source problem for the equation

where and its limit case, when ,

The existence and uniqueness results will be derived. Moreover, the stability theorem is also proved.

In Given problems, we impose the condition that should be strictly positive to avoid technical issues when obtaining estimates. However, the problem can still be solved without this condition.

Remark 3.1.1 Results of this section are published in the “Journal of Inverse and Ill-Posed Problems” in 2022 [13, р. 393-407] (joint work with D. Serikbaev and N. Tokmagambetov).

Let us introduce the Sobolev space on , as following

with norm

3.1.1 This subsection deals with the Cauchy problems for the nonhomogeneous heat equation with the Caputo fractional derivative , and its limit case , associated with the Dunkl operator (1.2.2).

Problem 3.1.2 Let . Find the function satisfying the equation

(3.1.1)

in the domain , under the initial condition

(3.1.2)

where and are sufficiently smooth functions, is the Dunkl operator (1.2.2).

A generalized solution of Problem 3.1.2 is a function satisfying the equation (3.1.1).

Theorem 3.1.3 Let , and . Then there exists a unique generalized solution of Problem 3.1.2. Moreover, it is given by the expression

where is the Dunkl kernel (1.2.6) and and are Mittag-Leffler functions.

Proof. Let . We are looking for a solution of Problem 3.1.2 from and , so we able to apply the Dunkl transform (1.2.17) according to the variable to the equation (3.1.1) and the initial condition (3.1.2). Then it gives us

(3.1.3)

and

(3.1.4)

for all where is an unknown function. Then by solving the equation (3.1.3) under the initial condition (3.1.4) (see [31, p. 231] and [84), we obtain

(3.1.5)

where and are Mittag-Leffler functions. Consequently, one obtains the solution of Problem 3.1.2, given by

by using the inverse Dunkl transform to (3.1.5), where is the Dunkl kernel (1.2.6).

We have following useful formula

Indeed, it follows from

Then, taking into account this and integrating by parts, one obtains

Thus,

Let us also introduce following useful inequalities. For every , we have:

1. if , then and .

2. if , then and .

Let and . Then for the function we have the following estimates

where denotes for some positive constant independent of and . Thus,

and . Now, let us introduce also following useful inequalities. For every , we have:

1. if , then and .

2. if , then and .

Then for , we obtain

Consequently, it gives us

Then using Definition 1.3.4, we obtain . The existence is proved.

Suppose that there are two solutions and of Problem 3.1.2. Denote

Then the function satisfies the equation

(3.1.6)

and the condition

(3.1.7)

Then by applying the Dunkl transform (1.2.17) to the equation (3.1.6) and the condition (3.1.7), one obtains

According to our analysis, above problem has a unique solution for all . Hence, using Theorem 1.2.22 (Plancherel theorem) we obtain

for all . The uniqueness of the solution of Problem 3.1.2 is proved. The theorem is fully proved.

Now, let us consider a limit case of Problem 3.1.2, when . When , instead of the Caputo fractional derivative we obtain usual partial derivative . Then Problem 3.1.2 turns into the following problem:

Problem 3.1.4 We aim to find a function satisfying the equation

under the condition

Theorem 3.1.5 Let and . Then Problem 3.1.4 has a unique generalized solution given by

(3.1.8)

Remark 3.1.6 The solution agrees with the solution of the Problem 3.1.2 expressed by

We obtain (3.1.8), when (because ).

Proof of Theorem 3.1.5. Let us first prove the existence of the solution of Problem 3.1.4. By using the Dunkl transform (1.2.17) to the Problem 3.1.4 according to the variable , we obtain the ordinary differential equation

(3.1.9)

with initial condition

(3.1.10)

respect to the variable . The general solution of the equation (3.1.9) can be written as

where the function is unknown. After using (3.1.10), one has

Then applying the inverse Dunkl transform , we obtain

Let and . Then we have

Thus,

Now estimating the function by

one gets

The existence is proved.

Now, we will prove the uniqueness of the solution of Problem 3.1.4. Let us suppose that and are two different solutions of Problem 3.1.4. Then is the solution to the following problem:

Then applying same technique, we able to see that the above problem has only trivial solution for all , showing the uniqueness of the solutions of the Problem 3.1.4. The theorem is fully proved.

3.1.2 In this subsection, we deal with an inverse source problem concerning the time-fractional heat equation with the Caputo fractional derivative , generated by the Dunkl operator and its limit case .

Problem 3.1.7 Let . Find a couple of functions satisfying the equation

(3.1.11)

in the domain , under the initial condition

and the over-determination condition

where and are sufficiently smooth functions, is the Dunkl operator (1.2.2).

A generalized solution of Problem 3.1.7 is a pair of functions , where

Theorem 3.1.8 Let and . Then a generalized solution of Problem 3.1.7 exists and is unique. And, moreover, it can be written by the expressions

and

where is the classical Mittag-Leffler function.

Proof. By applying the Dunkl transform (1.2.17), according to the variable , to Problem 3.1.7, one obtains

(3.1.12)

(3.1.13)

(3.1.14)

where and are unknown. The solution of the equation (3.1.12) ([37, р. 1-24]) is of the following form

(3.1.15)

where the constants and are unknown. To find these constants, we will use conditions (3.1.13) and (3.1.14). Hence, for we have

Thus,

Then the unknown can by represented as

Consequently, by substituting and into (3.1.15), we arrive at

and

Finally, Problem 3.1.7 is formally solved and a couple of functions are given by

and

by using the inverse Dunkl transform .

The inequalities (1.3.1) and (1.3.2) lead the following inequalities

Let , then we have the following estimates

Thus, . For every fixed we obtain

Hence, it gives . Rewriting the equation (3.1.12)

we arrive at

Consequently, we obtain

and . The existence is proved.

Suppose that there are two solutions and of Problem 3.1.7. Denote

and

Then the functions and satisfy the equation (3.1.11) and homogeneous conditions

(3.1.16)

Then by applying the Dunkl transform (1.2.17) to the equation (3.1.11) and the conditions (3.1.16), we obtain

Consequently, we have , for all , so Theorem 1.2.22 (Plancherel theorem) leads

Hence, , for all and the uniqueness of the solutions of Problem 3.1.7 is proved. The theorem is fully proved.

Now, let us consider limit case of the Problem 3.1.7, when . It is formulated as following:

Problem 3.1.9 We aim to find a couple of functions satisfying the equation

under the initial condition

and the over-determination condition

where and are sufficiently smooth functions, is the Dunkl operator (1.2.2).

Theorem 3.1.10 Assume that . Then Problem 3.1.9 has a unique generalized solution , where and . Moreover, they can be represented in the forms

and

Remark 3.1.11 The solution of Problem 3.1.9 agrees with the solution of Problem 3.1.7, when .

Proof of Theorem 3.1.10. Let us first prove the existence part. Applying the Dunkl transform (1.2.17), according to the variable , to Problem 3.1.9, we obtain

(3.1.17)

(3.1.18)

(3.1.19)

Then for every the general solution of the ordinary differential equation (3.1.17) is

(3.1.20)

where the functions and are unknown. By using the conditions (3.1.18) and (3.1.19), one can find

Then can be represented as

(3.1.21)

Now, substituting the functions and into (3.1.20), one has

(3.1.22)

Finally, by using the inverse Dunkl transform to (3.1.21) and (3.1.22), we obtain the solution to Problem 3.1.9:

and

A simple calculation gives us the following helpful inequalities:

1. .

2. .

3. .

4. .

Let . Then for every fixed , one can be obtained

Thus, we arrive at . We also have

and . Rewriting the equation (3.1.17), we have

Hence, we obtain

The existence is proved.

The uniqueness of the solutions of Problem 3.1.9 can be shown by taking into account the property of Dunkl transform (the Plancherel theorem 1.2.22) and by seeing that the pair of functions can be uniquely determined.

3.1.3 In the above subsection, we showed the unique solvability of the inverse source problems (Problem 3.1.7 and Problem 3.1.9). This kind of tasks are in general ill-posed problems. Hence, it is sensitive to the change of data. Practically, our final time measurement contains errors. In the following statement we address the impact of this on a solution of Problem 3.1.7. The case of Problem 3.1.9 can be dealt in a similar way.

Theorem 3.1.12 Let and be solutions to Problem 3.1.7 corresponding to the data and its small perturbation , respectively. Then the solution of Problem 3.1.7 depends continuously on these data, namely, we have

and

Proof. From the definition of the Dunkl transform

we have

here we have used property of the integral. Then we arrive at

Consequently, we have

or

Writing in the form

we obtain

Thus,

It completes the proof.

**3.2 Time-fractional pseudo-parabolic equation with Caputo fractional derivative**

In this section, we are interested in studying the Cauchy problem for the time-fractional pseudo-parabolic equation

and the inverse source problem for the time-fractional pseudo-parabolic equation

associated with the Dunkl operator (1.2.2), where , and

In the case , the inverse source problem for the time-fractional pseudo-parabolic equation reduces to the inverse source problem for the time-fractional heat equation

which was considered in Section 3.1. So, in this paper we are interested considering only case when .

In Given problems, we impose the condition that should be strictly positive to avoid technical issues when obtaining estimates. However, the problem can still be solved without this condition.

Results of this section are published as a preprint in ”arxiv” in [42] in 2023 (joint work with N. Tokmagambetov).

3.2.1 Here we consider the direct problem stated as following.

Problem 3.2.1 Let . Our aim is to find the function satisfying the equation

(3.2.1)

under the initial condition

(3.2.2)

where and are sufficiently smooth functions.

The following theorem shows that Problem 3.2.1 has a unique generalized solution in the space , where .

Theorem 3.2.2 a) Let . Assume that and . Then a generalized solution of Problem 3.2.1 exists, is unique, and given by the expression

(3.2.3)

where and are the Mittag-Leffler functions.

b) Let . Assume that and . Then Problem 3.2.1 has a unique generalized solution, which is given by the expression (3.2.3).

Proof. (a) Solution of Problem 3.2.1 can be found by applying the Dunkl transform (1.2.17) to the equation (3.2.1) and the initial condition (3.2.2). Thus, we have

(3.2.4)

and

(3.2.5)

where is an unknown function. Let . For every fixed the equation (3.2.4) is an ordinary differential equation, respect to the variable , then by solving the equation (3.2.4) under the initial condition (3.2.5) (see [31, p. 231]), we obtain

(3.2.6)

where and are the Mittag-Leffler functions. Consequently, solution of Problem 3.2.1 is

here we have used the Fubini’s theorem and the inverse Dunkl transform to the expression (3.2.6).

By using the property

of the Mittag-Leffler function, we obtain

and

by using the Integration by Parts and the fact .

We assume that and . Then for the function we have the following estimates

Thus, we obtain

Now, for we have

Consequently, it gives us

Finally, using Definition 1.3.4 we obtain .

(b) Let . We assume that and . Then let us estimate the function as follows

Then

Let us estimate as follows

Thus,

The existence is proved.

Now, we are going to prove uniqueness of the solution. Suppose that there are two solutions and of Problem 3.2.1. Denote

Then the function is a solution of the problem

Then by applying the Dunkl transform (1.2.17), we obtain

Above equation has a trivial solution (see [31, p. 231]), i.e., for all . Then using Theorem 1.2.22 (Plancherel theorem) we have

and for all . Hence, uniqueness of the solution is proved. The theorem is proved.

3.2.2 Now, let us study the inverse source problem.

Problem 3.2.3 Let . Our aim to find the couple of functions satisfying the equation

(3.2.7)

under the initial condition

(3.2.8)

and the over-determination condition

(3.2.9)

where and is sufficiently smooth functions.

We assume that . Then generalized solution of Problem 3.2.3 is the pair of functions , where and .

Theorem 3.2.4 Let . We assume that . Then generalized solution of Problem 3.2.3 exists, is unique, and can be written by the expressions

and

Proof. We want to find a solution of Problem 3.2.3 by applying the Dunkl transform (1.2.17) to the equation (3.2.7) and the conditions (3.2.8) and (3.2.9). Then it gives us

(3.2.10)

and

(3.2.11)

(3.2.12)

where and are unknown. Let . Using expression (3.2.6) we can find solution of the equation (3.2.10) with initial condition (3.2.11), given by

(3.2.13)

where is unknown and is the Mittag-Leffler function. Then applying the condition (3.2.12) to the expression (3.2.13) we obtain

Thus, we can find unknown as following

(3.2.14)

Consequently, by substituting into (3.2.13), we have

(3.2.15)

Then, we obtain the solution of Problem 3.2.3, which is a couple of functions are given by the formulas

and

by using the inverse Dunkl transform for (3.2.14) and (3.2.15).

Let . Then for we have the following estimate

Thus, we have . For we obtain

Consequently, it gives

Rewriting the equation (3.2.10) as following

we have

Thus,

The existence is proved.

Now, we are going to prove uniqueness of the solution. Suppose that there are two solutions and of Problem 3.2.3. Denote

and

Then the functions and satisfy

Then by applying the Dunkl transform (1.2.17), we obtain

Via our calculation above, we can see that problem has a trivial solution, i.e., and for all , . Then using Theorem 1.2.22 (Plancherel theorem) we able to see that and for all , . Hence, uniqueness of the solution is proved. The theorem is proved.

3.2.3 In this subsection we study stability of the solution of the Problem 3.2.3.

Theorem 3.2.5 Let and be solutions to Problem 3.2.3 corresponding to the data and its small perturbation , respectively. Then the solution of Problem 3.2.3 depends continuously on these data, namely, we have

and

Proof. From definition of the Dunkl transform

we have

here we have used property of the integral. Then we have

Consequently, we obtain

By writing (3.2.14) in a form

we obtain

Thus,

We completed our proofs.

3.2.4 Here we test one sample case for the subject of the stability of the solution pair.

Let us consider the following inverse source problem for the pseudo-parabolic equation

with Dirichlet boundary conditions

where , , and . By applying Theorem 3.2.4, with and : our operator in time is and in space , we obtain the trivial solution pair:

Now we consider a perturbation of the previous problem in the following form

with conditions

where , (). Then using Theorem 3.2.4, one obtains solution of the perturbation problem, expressed by

(3.2.16)

and

(3.2.17)

Integrals (3.2.16) and (3.2.17) are converges absolutely, because

and

Indeed, integrals and can be represented as follows

and

In the following pictures you can find the graphics of the functions and for different epsilons (), figure 1, 2.

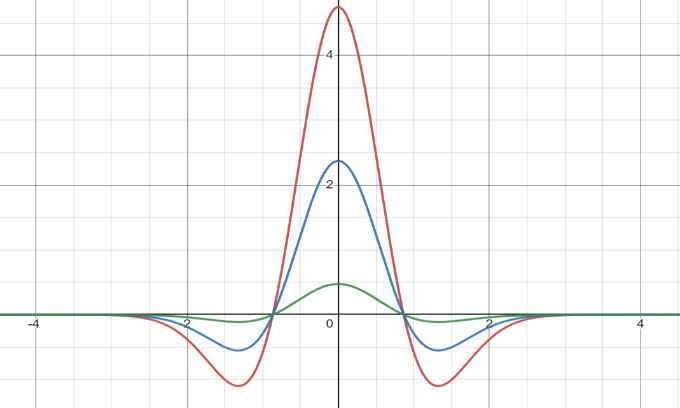


Figure 1 – The graph of the function , here we have used desmos.com. The red graph with , the blue graph with , and the green graph with .

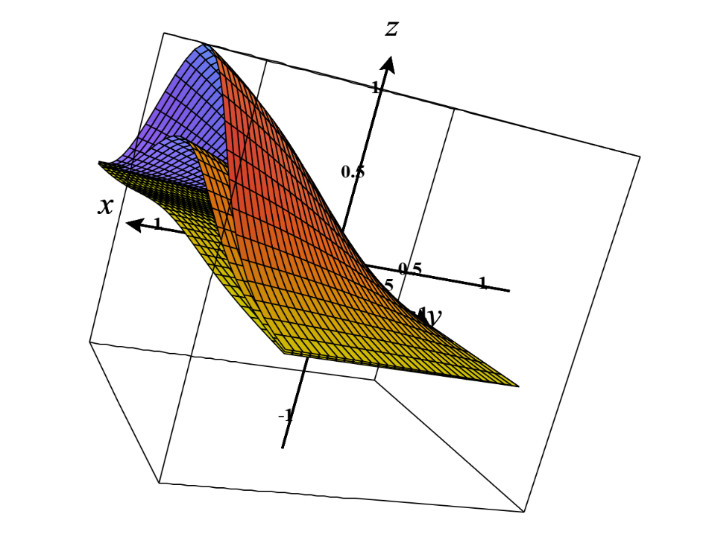


Figure 2 – The graph of the function , here we have used 3D Calc Plotter. The upper graph with , the middle graph with , and the lower graph with .

Now, let us calculate the following integrals:

since ,

and

since

According to the above computations we able to build a according table for different values of epsilon (), table 1.

Table 1 – Stability test

|  |  |  |  |
| --- | --- | --- | --- |
|  | 1 | 0.5 | 0.1 |
|  | 1.73205 | 0.86602 | 0.173205 |
|  | 2.74006 | 1.37003 | 0.274006 |
|  | 1.73205 | 0.86602 | 0.173205 |

Conclusion. In this subsection, we have considered one inverse source problem and defined its solution using our calculus developed in this paper. Then to examine stability of its solution, we considered perturbation problem. Table 1 shows that solution of the inverse source problem is stable regarding to the small changes of the dates.

**3.3 Time-fractional heat equation with bi-ordinal Hilfer fractional derivative**

In this section, we study the inverse source problem for the heat equation

where (, ) is the bi-ordinal Hilfer fractional derivative, () is the Dunkl operator, is a given function and and are unknown functions, which we should define. The necessary information about the bi-ordinal Hilfer fractional derivative and the Dunkl operator can be found in the preliminaries.

Definition 3.3.1 Let us define the space , as following

and norm in this space is defined by

Definition 3.3.2 We consider the spaces and , with the norms

and

respectively.

3.3.1 We consider the following Cauchy problem (Problem 3.3.3). The purpose of considering Problem 3.3.3 is to help to solve the inverse source problem, because when dealing with the inverse source problem, we need to know the unique solution of the direct problem.

Let us consider the equation

(3.3.1)

where , , and , on the domain .

Remark 3.3.3 In a case , the bi-ordinal Hilfer fractional derivative gives the Caputo fractional derivative, so results for this problem coincides with results of first section.

Definition 3.3.4 We will call the function u a regular solution if it satisfies regularity conditions

, and

and the equation (3.3.1) for all , where .

Problem 3.3.5. Our aim is to find a regular solution of the equation (3.3.1) on the domain , which satisfying the initial condition

(3.3.2)

where is given functions.

Theorem 3.3.6 We assume that and is finite for every fixed , , and . Then Problem 3.3.5 has a unique regular solution and . Moreover, it has an expression

where .

Proof. The existence of the solution. We assume that . Then we can interpret function as a tempered distribution and apply the Dunkl transform (1.2.17). Thus, we obtain ordinary differential equation

(3.3.3)

respect to the variable with an initial condition

(3.3.4)

for every fixed . After using Remark 1.3.8, we are able to rewrite the equation (3.3.3) as

where and . Then applying the operator we obtain

or

Here we suppose that . Therefore, we have

or

(3.3.5)

For every fixed the equation (3.3.5) is the linear Volterra integral equation of the second kind. Solution of the Volterra equation can be found by using the method of successive approximations ([31, p. 222-223]) and it has a form

(3.3.6)

The function , expressed by (3.3.6), is a solution of the problem (3.3.3)-(3.3.4) for every . In this stage let us check some statements about which we assumed before. So, we do following computations

and

Then we have . Now, applying the inverse Dunkl transform to the we obtain solution of Problem 3.3.5, which has a form

In the beginning of our proof, we made the assumption that . Now let us proof that it is correct, indeed

Then using Hölder’s inequality and positivity of the expressions and , we obtain

After applying Fubini’s theorem and supposing , we have

Hence, , whenever . Now, using the same computations and assumptions as previously, we are able to calculate

Hence, , whenever . After taking the maximum on variable , where , on both sides of the last inequality and using Fubini’s theorem we are able to calculate

Finally, let us show that . Hence, we need to calculate

Moreover, we have

Uniqueness of the direct problem. Let there be two solutions and of Problem 3.3.5. After we set . Then we obtain

Hence, Theorem 3.3.6 gives us unique solution of the above problem for all and , which implies , thanks to Plancherel theorem (Theorem 1.2.22). The proof is complete.

3.3.2 In this subsection, we consider the main problem of our section, the inverse source problem generated by the bi-ordinal Hilfer operator (, ) and Dunkl operator (). We prove Theorem 3.3.9, where we show unique solvability of Problem 3.3.7.

Problem 3.3.7 Let , , and . Our aim is to find a solution pair of the inverse source problem

(3.3.7)

on the domain , satisfying the conditions

(3.3.8)

and

where the functions , and are given functions.

Remark 3.3.8 If and , then the results for this problem coincides with results of first section.

Theorem 3.3.9 Let . We assume that and

is a finite well-defined nonzero number for every and , and . Then Problem 3.3.7 has a solution pair , where is a regular solution, which are and with , and expressed by

and

Proof. The existence of the solution. As in previous section we suppose that . Then we can interpret functions and as tempered distributions and apply the Dunkl transform (1.2.17). Hence, we obtain ordinary differential equation

(3.3.9)

respect to the variable with the conditions

(3.3.10)

and

for every fixed . The analysis of Subsection 3.3.1 gives us the unique solution of the Cauchy problem (3.3.9)-(3.3.10) and it has a form

(3.3.11)

Then applying the condition to the (3.3.11) we have

From we define

(3.3.12)

After substituting into (3.3.11) we obtain

(3.3.13)

After applying the inverse Dunkl transform to the formulas (3.3.12) and (3.3.13) we obtain solution of Problem 3.3.7, which are expressed by

and

Now let us show that , whenever . For this, we need to calculate

Then the following computations

and

gives us , whenever and

Now, using previous computations we have

and

Thus, , whenever . Then we obtain

and

The uniqueness of the solution. Let there are two solutions and of Problem 3.3.7. After we set and . Then we obtain

Hence, Theorem 3.3.9 gives us unique solution of the above problem and for all and , which implies and , thanks to Theorem 1.2.22 (Plancherel theorem). The proof is complete.

3.3.3 In this subsection, we show stability of Problem 3.3.7.

Theorem 3.3.10 Let and be solutions to Problem 3.3.7 corresponding to the data and its small perturbation , respectively. Then the solution of Problem 3.3.7 depends continuously on these data. Moreover, we obtain

and

Proof. Let and be solutions to Problem 3.3.7 corresponding to the data and , respectively, which satisfy Theorem 3.3.9. Then we are able to calculate

and

Then using previous computations, we obtain

and

Also, for , we can obtain the same estimates

Hence, we have

Remark 3.3.11 Readers also can be familiar with recent works connected with direct and inverse problems with bi-ordinal Hilfer fractional derivative from [85, 86].

**CONCLUSION**

In this thesis, we develop analysis of pseudo-differential operators and considered some inverse source problems generated by the Dunkl operators on the real line. Let us review the obtained results in this thesis:

In Chapter 3, we considered pseudo-differential operators generated by the Dunkl operator. We proved that these operators are continuous linear operators on . We defined amplitude, adjoint and transpose operators and proved that they are also continuous linear operators on . Then we studied distributional and convolution kernels of the pseudo-differential operators generated by the Dunkl operator and proved certain properties of them. In the last section, we considered boundedness results of the pseudo-differential operators under some assumptions. The results of this chapter are unpublished, and we hope that some papers will follow after the PhD defense.

We can envision two possible continuations of the results from Chapter 3:

1. Can we developer a symbolic calculus for this analysis? To answer this question, we may start from revising [87].

2. Can we extend obtained results to the higher dimensions, especially in ? Also, we can try to find some application to this analysis.

In Chapter 4, we studied some inverse source problems generated by the Dunkl operator. More precisely, inverse source problems for heat and pseudo-parabolic equations with Caputo and bi-ordinal Hilfer fractional differential operators, generated by the Dunkl operator. We obtained well-posedness results in the sense of Hadamard. First two parts of this chapter is based on our published work [13] (joint work with D. Serikbaev and N. Tokmagambetov) and as a preprint in ”arxiv” in [42] in 2023 (joint work with N. Tokmagambetov).

All problems considered in Chapter 4 are linear and have constant coefficients. A possible continuation of the results in this chapter is to explore linear problems with variable coefficients and to extend the analysis to non-linear problems. Additionally, these considerations can be extended to higher dimensions, especially in . For this kind of problems, it may be useful to consider discrete Dunkl analysis, so questions raises in this direction as well.

**REFERENCES**

1 Dunkl C.F. Reflection groups and orthogonal polynomials on the sphere // Mathematische Zeitschrift. – 1988. – Vol. 197, №1. – P. 33-60.

2 Dunkl C.F. Differential-difference operators associated to reflection group // Transactions of the American Mathematical Society. – 1989. – Vol. 311, №1. – P. 167-183.

3 Dunkl C.F. Operators commuting with Coxeter group actions on polynomials // In book: Invariant Theory and Tableaux. – Springer, 1990, – P. 107-117.

4 Dunkl C. F. Integral kernels with reflection group invariant // Canadian Journal of Mathematics. – 1991. – Vol. 43, №6. – P. 1213-1227.

5 Dunkl C.F. Hankel Transforms Associated to Finite Reflection Groups // Contemporary Mathematics. – 1992. – Vol. 138. – P. 123-138.

6 de Jeu M.F.E. The Dunkl transform // Inventiones mathematicae. – 1993. – Vol. 113, №1. – P. 147-162.

7 Dachraoui A. Pseudodifferential-difference operators associated with Dunkl operators // Integral Transforms and Special Functions. – 2001. Vol. 12, №2. – P. 161-178.

8 Abdelkefi C., Amri B., Sifi M. Pseudo-differential operator associated with the Dunkl operator // Differential and Integral Equations. – 2007. – Vol. 20. – P. 1035-1051.

9 Amri B., Mustapha S., Sifi M. On the boundedness of pseudo-differential operators associated with the Dunkl transform on the real line, Advances in Pure and Applied Mathematics. – 2010. – Vol. 2. – P. 89-107.

10 Rösler M. Generalized Hermite Polynomials and the Heat Equation for Dunkl Operators // Communications in Mathematical Physics. – 1998. – Vol. 192, №3. – P. 519-542.

11 Mejjaoli H. Dunkl heat semigroup and applications // Applicable Analysis. – 2013. – Vol. 92, №9. – P. 1980-2007.

12 Mejjaoli H. Generalized heat equation and applications // Integral Transforms and Special Functions. – 2014. – Vol. 25, №1. – P. 15-33.

13 Bekbolat B., Serikbaev D., Tokmagambetov N. Direct and inverse problems for time-fractional heat equation generated by Dunkl operator // Journal of Inverse and Ill-Posed Problems. – 2023. – Vol. 31, №3. – P. 393-408.

14 Bekbolat B., Kassymov A., Tokmagambetov N. Blow-up of solutions of nonlinear heat equation with hypoelliptic operators on graded Lie groups // Complex Analysis and Operator Theory. – 2019. – Vol. 13, №7. – P. 3347-3357.

15 Bekbolat B., Tokmagambetov N. Well-posedness results for the wave equation generated by the Bessel operator // Bulletin of the Karaganda University. – 2021. – Vol. 101, №1. – P. 11-16.

16 Bekbolat B., Nurakhmetov D.B., Tokmagambetov N. et al. On the minimality of systems of root functions of the Laplace operator in the punctured domain // News of the national academy of sciences of the republic of Kazakhstan. – 2019. – Vol. 4, №326. – P. 92-109.

17 Bekbolat B., Kanguzhin B., Tokmagambetov N. To the question of a multipoint mixed boundary value problem for a wave equation // News of the national academy of sciences of the republic of Kazakhstan. – 2019. – Vol. 4, №326. – P. 76-82.

18 Bekbolat B., Tokmagambetov N. On a boundedness result of non-toroidal pseudo differential operators // International Journal of Mathematics and Physics. – 2018. – Vol. 9, №2. – P. 50-55.

19 Ruzhansky M., Turunen V. Pseudo-Differential Operators and Symmetries: Background Analysis and Advanced Topics. Pseudo-Differential Operators. Theory and Applications 2. – Basel: Birkhäuser, 2010. – 724 p.

20 Gask H. A proof of Schwartz’s kernel theorem // Mathematica Scandinavica. – 1960. – Vol. 8. – P. 327-332.

21 Rösler M. Dunkl Operators: Theory and Applications // In book: Orthogonal Polynomials and Special Functions. – Springer; Berlin, 2003. – P. 93-135.

22 Anker J.-Ph. An introduction to Dunkl theory and its analytic aspects // In book: Analytic, Algebraic and Geometric Aspects of Differential Equations. – Chem: Birkhäuser, 2017. – P. 3-41.

23 Betancor J.J., Sifi M., Trimèche K. Hypercyclic and chaotic convolution operators associated with the Dunkl operator on // Acta Mathematica Hungarica. – 2005. – Vol. 106, №1-2. – P. 101-116.

24 Trimèche K. Generalized Harmonic Analysis and Wavelets Packets. – Boca Raton, 2001. – 320 p.

25 Rosenblum M. Generalized Hermite polynomials and the Bose-like oscillator calculus // Operator Theory Advances and Applications. – 1994. – Vol. 73. – P. 369-396.

26 Функции Бесселя: учеб.-метод. пос. / сост. В.И. Зубов. – М.: МФТИ, 2007 – 51 с.

27 Said S.B., Orsted B. The wave equations for Dunkl operators // Indagationes Mathematicae. – 2005. – Vol. 16, – P. 351-391.

28 Rösler M. Bessel-type signed hypergroups on // In book: Probability Measures on Groups and Related Structures XI. – Singapore: World Scientific, 1995. – P. 292-304.

29 Soltani F. -Fourier multipliers for the Dunkl operator on the real line // Journal of Functional Analysis. – 2004. – Vol. 209. – P. 16-35.

30 Mourou M.A. Taylor series associated with a differential-difference operator on the real line // Journal of Computational and Applied Mathematics. – 2003. – Vol. 153, №1-2. – P. 343-354.

31 Kilbas A.A., Srivastava H.M., Trujillo J.J. Theory and Applications of Fractional Differential Equations. – Amsterdam: Elsevier, 2006. – 530 p.

32 de Carvalho-Neto P.M. et al. Conditions for the absence of blowing up solutions to fractional differential equations // Acta Applicandae Mathematicae. – 2018. – Vol. 154. – P. 15-29.

33 Hilfer R. Fractional time Evolution // In book: Applications of Fractional Calculus in Physics. –Singapore: World Sci., 2000. – P. 89-130.

34 Hilfer R. Experimental evedence for fractional time evolution in glass forming materials // J. Chem. Phys*.* – 2002. – Vol. 284. – P. 399-408.

35 Hilfer R., Luchko Y., Tomovski Z. Operational method for solution of the fractional differential equations with the generalized Riemann-Liouville fractional derivatives // Frac. Calc. Appl. Anal*.* – 2009. – Vol. 12. – P. 299-318.

36 Toshtemirov B. Direct and inverse problems for singular partial differential equations with fractional order integral-differential operators: thes. … doc. PhD. – Tashkent: Ghent University and V.I. Romanovskiy institute of mathematics, 2022. – 167 p.

37 Simon T. Comparing Fréchet and positive stable laws // Electronic Journal of Probability. – 2014. – Vol. 19. – P. 1-25.

38 Podlubny I. Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. – San Diego: Academic Press, 1999. – 340 p.

39 Folland G.B. Introduction to partial differential equations. – Ed. 2nd. – Princeto: Princeton University Press, 1995. – 324 p.

40 Prasad A., Singh M.K. Composition of Pseudo-Differential Operators Associated with Jacobi Differential Operator // Proceedings of the National Academy of Sciences, India Section A: Physical Sciences. – 2019. – Vol. 89. – P. 509-516.

41 Prasad, A., Singh, M.K. Pseudo-differential operators associated with the Jacobi differential operator and Fourier-cosine wavelet transform // Asian-European Journal of Mathematics. – 2015. – Vol. 8, №1. – P. 1-16.

42 Bekbolat B., Tokmagambetov N. One inverse source problem generated by the Dunkl operator // <https://arxiv.org/abs/2308.01232>. 10.10.2023.

43 B. Bekbolat, N. Tokmagambetov. Cauchy problem for the Jacobi fractional heat equation // Kazakh Mathematical Journal. – 2021. – Vol. 21, №3. – P. 16-26.

44 Rundell W., Colton D.L. Determination of an unknown nonhomogeneous term in a linear partial differential equation from overspecified boundary data // Applicable Analysis. – 1980. – Vol. 10, №3. – P. 231-242.

45 Prilepko A.I., Tikhonov I.V. Uniqueness of a solution of the inverse problem for the evolution equation and application to the transport equation // Mathematical Notes. – 1992. – Vol. 51. – P. 158-165.

46 Bushuyev I. Global uniqueness for inverse parabolic problems with final observation // Inverse Problems. – 1995. – Vol. 11. – P. L11-L16.

47 Tikhonov I.V., Eidelman Yu.S. An inverse problem for a differential equation in a Banach space and distribution of zeros of an entire Mittag-Leffler function // Differential Equations. – 2002. – Vol. 38, №5. – P. 669-677.

48 Choulli M., Yamamoto M. Conditional stability in determining a heat source // Journal of Inverse and Ill-Posed Problems. – 2004. Vol. 12, №3. – P. 233-243.

49 Yaman M., Gözükızıl O.F. Asymptotic behaviour of the solutions of inverse problems for pseudo-parabolic equations // Applied Mathematics and Computation. – 2004. – Vol. 154. – P. 69–74.

50 Sakamoto K., Yamamoto M. Inverse source problem with a final overdetermination for a fractional diffusion equation // Mathematical control and related fields. – 2011. – Vol. 1, №4. – P. 509-518.

51 Yaman M. Blow-up solution and stability to an inverse problem for a pseudo-parabolic equation // Journal of Inequalities and Applications. – 2012. – Vol. 2012. – P. 274-1-274-8.

52 Slodička M. A source identification problem in linear parabolic problems: A semigroup approach // Journal of Inverse and Ill-Posed Problems. – 2013. – Vol. 21. – P. 579-600.

53 Slodička M., Šišková K. An inverse source problem in a semilinear time-fractional diffusion equation // Computers and Mathematics with Applications. – 2016. – Vol. 72. – P. 1655-1669.

54 Ruzhansky M., Tokmagambetov N., Torebek B.T. Inverse source problems for positive operators. I: Hypoelliptic diffusion and subdiffusion equations // Journal of Inverse and Ill-Posed Problems. – 2019. – Vol. 27, №6. – P. 891-911.

55 Ruzhansky M., Serikbaev D., Torebek B.T. et al. Direct and inverse problems for time-fractional pseudo-parabolic equations // Quaestiones Mathematicae. – 2022. – Vol. 45, №7. – P. 1071-1089.

56 Cannon J.R., Du Chateau P. Structural identification of an unknown source term in a heat equation // Inverse Problems. – 1998. – Vol. 14, №3. – P. 535-551.

57 Slodička M. Uniqueness for an inverse source problem of determining a space-dependent source in a non-autonomous time-fractional diffusion equation // Fractional Calculus and Applied Analysis. – 2020. – Vol. 23, №6. – P. 1702-1711.

58 Slodička M. Uniqueness for an inverse source problem of determining a space dependent source in a non-autonomous parabolic equation // Applied Mathematics Letters. – 2020. – Vol. 107. – P. 1702-1711.

59 Aitzhanov S.E., Zhanuzakova D.T. Behavior of solutions to an inverse problem for a quasilinear parabolic equation // Siberian Electronic Mathematical Reports. – 2019. – Vol. 16. – P. 1393-1409.

60 Lyubanova A.S., Tani A. An inverse problem for pseudo-parabolic equation of filtration: the existence, uniqueness and regularity // Applicable Analysis. – 2011. – Vol. 90, №10. – P. 1557-1571.

61 Lyubanova A.S., Tani A. On inverse problems for pseudoparabolic and parabolic equations of filtration // Inverse Problems in Science and Engineering. – 2011. – Vol. 19, №7. – P. 1023-1042.

62 Khompysh, K., Shakir A.G. An inverse source problem for a nonlinear pseudoparabolic equation with p-Laplacian diffusion and damping term // Quaestiones Mathematicae. – 2023. – Vol. 46, №9. – P. 1889-1914.

63 Shakir A., Kabidoldanova A., Khompysh K. Solvability of a nonlinear inverse problem for a pseudoparabolic equation with p-laplacian // KazNU Bulletin. Mathematics, Mechanics, Computer Science Series. – 2021. – Vol. 110, №2. – P. 35-46.

64 Khompysh K., Shakir A. The inverse problem for determining the right part of the pseudo-parabolic equation // KazNU Bulletin. Mathematics, Mechanics, Computer Science Series. – 2020. – Vol. 105, №1. – P. 87-98.

65 Khompysh K. Inverse Problem for 1D Pseudo-parabolic Equation // Springer Proceedings in Mathematics and Statistics. – 2017. – Vol. 216. – P. 382-387.

66 Antontsev S.N., Aitzhanov S.E., Ashurova G.R. An inverse problem for the pseudo-parabolic equation with p-laplacian // Evolution Equations and Control Theory. – 2022. – Vol. 11, №2. – P. 399-414.

67 Aitzhanov S.E., Ashurova G.R., Zhalgassova K.A. Identification of the right hand side of a quasilinear pseudoparabolic equation with memory term // KazNU Bulletin. Mathematics, Mechanics, Computer Science Series. – 2021. – Vol. 110, №2. – P. 47-63.

68 Cheng J., Nakagawa J., Yamamoto M., Yamazaki T. Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation // Inverse Problems. – 2009. – Vol. 25. – P. 115002-1-115002-28.

69 Jin B., Rundell W. A tutorial on inverse problems for anomalous diffusion processes // Inverse Problems. – 2015. – Vol. 31, №3. – P. 035003-1-035003-40.

70 Kaliev I.A., Sabitova M.M. Problems of determining the temperature and density of heat sources from the initial and final temperatures // Journal of Applied and Industrial Mathematics. – 2010. – Vol. 4, №3. – P. 332-339.

71 Kirane M., Samet B., Torebek B.T. Determination of an unknown source term temperature distribution for the sub-diffusion equation at the initial and final data // Electronic Journal of Differential Equations. – 2017. – Vol. 2017. – P. 1-13.

72 Orazov I., Sadybekov M.A. One nonlocal problem of determination of the temperature and density of heat sources // Russian Mathematics. – 2012. – Vol. 56, №2. – P. 60-64.

73 Orazov I., Sadybekov M.A. On a class of problems of determining the temperature and density of heat sources given initial and final temperature // Siberian Mathematical Journal. – 2012. – Vol. 53, №1. – P. 146-151.

74 Slodička M., Šišková K., Van Bockstal K. Uniqueness for an inverse source problemof determining a space dependent source in a time-fractional diffusion equation // Applied Mathematics Letters. – 2019. – Vol. 91. – P. 15-21.

75 Torebek B.T., Tapdigoglu R. Some inverse problems for the nonlocal heat equation with Caputo fractional derivative // Mathematical Methods in the Applied Sciences. – 2017. – Vol. 40, №18. – P. 6468-6479.

76 Wang W., Yamamoto M., Han B. Numerical method in reproducing kernel space for an inverse source problem for the fractional diffusion equation // Inverse Problems. – 2013. – Vol. 29. – P. 095009.

77 Ashurov R., Kadirkulov B., Ergashev O. Inverse Problem of Bitsadze–Samarskii Type for a Two-Dimensional Parabolic Equation of Fractional Order // Journal of Mathematical Sciences (United States). – 2023. – Vol 274, №2. – P. 172-185.

78 Al-Salti N., Karimov E. Inverse Source Problems for Degenerate Time-Fractional PDE // Progress in Fractional Differentiation and Applications. – 2022. – Vol. 8, №1. – P. 39-52.

79 Dib F., Kirane M., An Inverse Source Problem for a Two Terms Time-fractional Diffusion Equation // Boletim da Sociedade Paranaense de Matematica. – 2022. – Vol. 40. – P. 1-15.

80 El Hamidi A., Kirane M., Tfayli A. An Inverse Problem for a Non-Homogeneous Time-Space Fractional Equation // Mathematics. – 2022. – Vol. 10, №15. – P. 2586-1-2586-17.

81 Ilyas A., Malik S.A. An Inverse Source Problem for Anomalous Diffusion Equation with Generalized Fractional Derivative in Time // Acta Applicandae Mathematicae. – 2022. – Vol. 181, №1. – P. 15-1-15-16.

82 Van Diejen J.F., Vinet L. Calogero-Sutherland-Moser Models. – Berlin: Springer, 2000. – 586 p.

83 Chouchene F., Gallardo L., Mili M. The heat semigroup for the Jacobi-Dunkl operator and the related Markov processes // Potential Analysis. – 2006. – Vol. 25, №2. – P. 103-119.

84 Luchko Y., Gorenflo R. An operational method for solving fractional differential equations with the Caputo derivatives // Acta Mathematica Vietnamica. – 1999. – Vol. 24. – P. 207-233.

85 Karimov E., Ruzhansky M., Toshtemirov B. Solvability of the boundary-value problem for a mixed equation involving hyper-Bessel fractional differential operator and bi-ordinal Hilfer fractional derivative // Mathematical Methods in the Applied Sciences. – 2023. – Vol. 46, №1. – P. 54-70.

86 Toshtemirov B. Direct and inverse source problem for 2D Landau Hamiltonian operator // [https://www.degruyter.com/document/doi/10.1515/gmj-2023-2059. 25.10.2023](https://www.degruyter.com/document/doi/10.1515/gmj-2023-2059.%2025.10.2023).

87. Halbout G., Tang X. Dunkl operator and quantization of -singularity // J. reine angew. Math. – 2012. – Vol. 673. – P. 209-235.