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**Properties Preservation of Expansion of Models of NIP Theories**

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**NORMATIVE REFERENCES**

There are used next references for standards in the present thesis:

SOSE RK 5.04.034-2011. State obligatory standard of education. Postgraduate education. Doctoral studies.

State standard 7.32-2001. Report on scientific-research work. Structure and rules of presentation.

State Standard 7.1-2003. Bibliographic record. Bibliographic description. General requirements and rules.

**DEFINITIONS**

There are used next designators with the corresponding definitions in the dissertation. At the beginning we give the main model-theoretic determinations and designators. [1-7]. We will introduce more complicated notions in further sections.

A **signature** or a **language** Σ (or L) consists of the following symbols:

1) functional symbols *fi*

2) relational symbols Ri

3) constant symbols ci

An L-structures is defined as follows: an **universe** N of a structure with *fN: ,* and *.*

We say that A is *convex* in for subset of a linearly ordered structure if it follows where for any such that .

Let be structures of a signature . We say that is a *substructure* of (denoted by ) if for any quantifier free formula and for any the next holds:

Suppose , are structures of a signature . is called an  *elementary submodel* of (notation is) if for every formula and for all the next holds:

We say that a formula is *convex to the right* where is a formula in a linearly ordered structure if

Suppose is a linearly ordered structure. We say that *-cut* in in case of the splitting of into two convex subsets and (; ). If has a supremum or has a infimum in , thereat the -cut is said to be  *rational*. Else, the -cut is *irrational*. Sometimes, by -cut we mean the next a number of formulas:

Suppose . We say that a -definable formula is  *-preserving* or *-stable* if for every there are , that

In occation, where is right (left) endpoint of set we say that formula is convex to the left (right).

If () formula which is the greatest -preserving convex to the left (right) 2-B-formula thereat is  *semi-quasisolitary to the left (right)* where is a non-algebraic type. It is called that . It means that for every -preserving convex to the left formula , and for every

If type is semi-quasisolitary to both sides then it is called to be *quasisolitary*.

Let be quasisolitary. When the greatest convex to the left and to the right formula then it is called that is *solitary*

**NOTATIONS AND ABBREVIATIONS**

|  |  |  |
| --- | --- | --- |
|  |  | negation |
|  |  | universal quantifier |
|  |  | existential quantifier |
|  |  | implication |
| , |  | disjunction and conjunction |
|  |  | elementary equivalence |
|  |  | structures |
|  |  | elementary substructure |
|  |  | if and only if |
|  |  | satisfaction in structure |
|  |  | languages |
|  |  | sets |
|  |  | elements of structures |
|  |  | tuples |
|  |  | universes of structures |
|  |  | elements of extensions of structures |
|  |  | types |
|  |  | formulas |
|  |  | all elements of set are less than any element of set |
|  |  | cardinality of a set |
|  |  | set of realizations of a formula (type ) in |
|  |  | neighbourhood, quasi-neighbourhood of in type |
|  |  | structure theory |
|  |  | set of any complete -type (over set ) of theory |
|  |  | number of models of cardinality of |
|  |  | type of over the set |
|  |  | definable (algebraic) closure of a set |
|  |  | relation of weak orthogonality of types |
|  |  | relation of almost orthogonality of types |

**INTRODUCTION**

**The research theme actuality.** At present expansion of models by new relations is one of the primary directions of reseach in theory of models which is part of the mathematical logic.

The underlying theme in model theory is to classify first order theories. The first approximation in classifying was Shelah’s notion of  *stable theory*. Which recently has broadened and nowadays includes NIP theories.

Theory is named to possess the independence property (IP), whenever it is possible to find a formula such that in every model of for each , a family of tuples exists, to such an extent that it is easy to find a tuple in such that for every subset of . If it doesn’t exist such formula, then is said to have NIP, that is not the independence property.

Important part of investigating complete theories is to examine specifications of new relations necessary and/or sufficient to change class of model of complete theory in new signature or preserve it. One of the most significant classes of complete theories in NIP theories along with stable theories are o-minimal theories and a wider class including o-minimal theories - weakly o-minimal theories. This classes of theories are in the main scope of exploration of this work.

Leading specialists in model theory, such as B.I. Zilber, E. Hrushovski, A. Nesin, B. Poizat, G.Cherlin, J. Baldwin, E. Bouscaren, A. Wilkie, Ch. Steinhorn, D. Macpherson, D. Marker, B. Baizhanov, S. Shelah, M. Benedikt, A.Pillay, have received profound results in different problems of expansions.

J.T. Baldwin and K.Holland found sufficient conditions that there is model complete theory behind every unary -categorical expansion of strongly minimal model. D. Macpherson, Ch. Steinhorn and D. Marker have verified that an expansion of weakly o-minimal structure by particular type of convex unary predicate preserves weak o-minimality. B.S. Baizhanov has resolved problem of the weakly o-minimal theories expansion using unary convex predicate [8]. B. Sh. Kulpeshov presented the concept of convexity rank and obtained a description of weakly o-minimal theories in terms of definable sets of one-types convexity. Thesis concerns different classes of expansions of finite convexity rank weakly o-minimal theories which is quite new class of complete NIP theories.

**The aim of the work** is to examine issues of certain properties preservation (like quite o-minimality, weak o-minimality, countable categoricity, model completeness, convexity rank and others) in the process of expansion of models.

**The objectives of the work** are the following:

- Investigate inquiries of certain properties preservation of expantion of models by unary predicates.

- Investigate inquiries of certain properties preservation of expantion of models by equivalence relations.

- Investigate inquiries of certain properties preservation of expantion of models by arbitrary binary predicates.

**The main states for the dissertation defense:**

- Touchstone for maintaning aleph-nought categoricity in the process of weakly o-minimal expansion of a non-1-indiscernible weakly o-minimal aleph-nought categorical theory of convexity rank 1 by every single binary predicate.

- Touchstone for maintaning aleph-nought categoricity for a weakly o-minimal expansion of a 1-indiscernible weakly o-minimal aleph-nought categorical theory of convexity rank 1 by every single binary predicate.

- Maintaining weak o-minimality when expanding a weakly o-minimal ordered group by an externally definable binary predicate.

- Touchstone for keeping weak o-minimality and countable categoricity (and the 1-indiscernibility in addition to this). It is in next case. A weakly o-minimal 1-indiscernible countably categorical theory which has finite convexity rank is considered. A model of the theory is expanded using an relation of equivalence splitting the universe into infinite number of infinite convex classes.

- Maintaining quite o-minimality, countable categoricity and convexity rank when expanding a model of a quite o-minimal countably categorical theory by a convex unary predicates family which is finite.

- Maintaining the convexity rank and the countable categoricity under expantion of a theory model where theory is countably categorical weakly o-minimal with finite convexity rank and expantion is made by a convex unary predicates family which is finite.

**The objects of research** are complete NIP theories (theories lacking the independence property) and models of NIP theories. In particular, NIP theories include weakly o-minimal theories and stable theories.

**The research subjects** models of NIP theories, their properties and properties of these models under expansion by unary or binary predicates or equivalence relations

**Methods of research:** In the dissertation we use Classical Model Theory methods (in particular, method of quantifier elimination), inclusive of the ones which have been developed in model theory since 1980’s and later. Among them we can note the methodology of investigating ordered structures, based on such notions as o-minimality and variants of o-minimality. In such cases it is typical to apply strict restrictions on sets definable by a formula which has the only free variable. Thus, if is a language which includes language , where is a linear order on an o-minimal structure and every single definable subset of the structure is quantifier-free in then we can consider the o-minimal structure as -structure. It gives pattern for other determination. We set another unknown language instead language . Then we regard -structures to an extent so that the -reduct is linear order or language of some stipulate type. And we want all definable subsets of structure are -definable (quantifier-free). It is possible to require for every model of the theory. Aside from that, we can note the methods of researching ordered structures developed in the last 20 years, such as describing models through analysis of behaviour of definable unary functions, the examine of models via systematization by convexity rank and others.

**Scientific novelty of the dissertation research.** Preservation of properties of expansion of models of complete theories such as NIP theory problem is unsettled at present. Contained in the survey classes of theories haven’t been researched on an considered expansions.

**Practical and theoretical research significance**. Systematization of complete NIP theories stem from researches in this area. We can apply anticipated conclusions on the essence of expansions to the fields, rings and groups theories.

**Dissertation thesis connection the with the another scientific investigation works**. The following list shows scientific projects within framework of which PhD dissertation was carried out. Scientific projects of the program of grant financing of fundamental researches of the MES of Republic of Kazakhstan: “Properties of types in dependent theories” (2015-2017, 5125/GF4), “Basic and derived objects for ordered and generating structural objects as well as elementary theories” (2018-2020, AP05132546) and “Conservative extensions, countable ordered models and closure operators” (2018-2020, AP05134992).

**Approbation of obtained results**: PhD thesis results are tested at many foreign and domestic international scientific conferences and seminars:

- The 12th International Conference School "Problems Allied to Universal Algebra and Model Theory" (2017, Russia, Erlagol);

- The Sixth Congress of the Turkic World Mathematical Society (Astana, 2017);

- ASL European Summer Meeting "Logic Colloquium" (Udine, Italy, 2018);

- International Conference "Mal’tsev Meeting" (Novosibirsk, Russia, Institute of Mathematics, 2017, 2018);

- ASL North American Annual Meeting (Macomb, USA, Western Illinois University, 2018);

- The 6-thWorld Congress and School on Universal Logic (2018, France, Vichy);

- The Sixteenth Asian Logic Conference (Nur-Sultan, 2019);

- Annual International April Mathematical Conference (Almaty, Institute of Mathematics and Mathematical Modeling, 2017, 2018, 2019, 2020).

**Publications**: Research findings of the PhD thesis were published in 20 works, including 3 articles published in journals having a non-zero impact factor according to international databases Web of Science and (or) Scopus; 4 papers published in domestic journals recommended by CCFES of the Ministry of Education and Science of Kazakhstan. Also 13 abstarcts were published in materials of international scientific conferences.

**Dissertation volume and structure.** The dissertation consists of next units and structural item: page of title, contents, prescriprive references, abbreviations, notations and definitions, introduction, five sections (historical review, expansions of models by unary predicates, expansions of models by equivalence relations, expansions of models by binary arbitrary predicates, external definability and model completeness), inference and links. Dissertation’s total number of pages equals 78. The work includes 94 references and four pictures.

**Dissertation work main content.** Section “Introduction” of the thesis includes the dissertation work aim, the research objectives of the thesis. It shows relevance of science research topic, scientific novelty of the research. There are the objects of investigation and the investigation subjects, primary states for the dissertation defence in the Introduction. The section explains practical and theoretical dissertation significance. There are described research methods. In the introduction there are given connection of the research thesis work with another scientific investigation works, approbation of obtained results, publications, as well as volume, structure and main content of the PhD thesis.

The first section describes the historical background and the present state of the model theory are under investigation.

The second section of the dissertation provides basic information and considers expansions of models by unary predicates.

The third section is devoted to expansions by equivalence relations of countably categorical, weakly ordered-minimal theories. Found a criterion for preserving countable categoricity and weak ordered-minimality.

In the fourth section of the dissertation considers arbitrary binary expansions of 1-indiscernible and non-1-indiscernible countably categorical models, weakly ordered-minimal theories of convexity rank 1.

The fifth section is focused on the class of externally definable expansions in the scope of preserving model completeness.

To fill out the section’s main result we also show different examples of expansions which does not preserve certain properties.

The conclusion lists and generalizes the key conclusions reached in the PhD thesis.

**1 HISTORICAL REVIEW**

Throughout all the history the establishment of theory of models may be related to a number of directions. model of signature of first order predicate logic solves the complete elementary theory of given model, , it means the set of sentences of signature , that holds in this model. We say *two models and of identical signature are elementary equivalent whensoever their elementary theories match or* . Investigations of elementary theory were droven by four main pathways of research:

- Elementary theory decidability;

- Quantity of non-isomorphic models of complete theories;

- Elementary theory axiomatizability;

- Expansion of models by new relations.

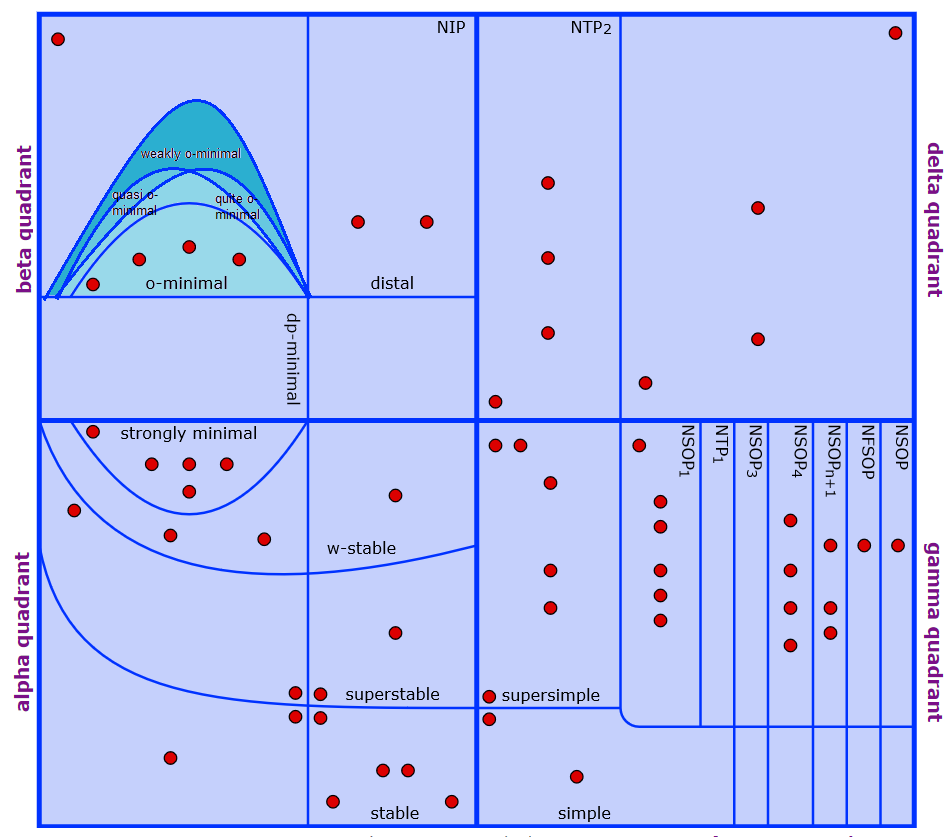
We state that a model is an expansion of a model if . The expansion is called an  **essential expansion** if forbye there exists an -ary -formula , to such an extent that the set is not definable over .

The dominant problem in this field of model theory is: for requisite features of the initial model, to obtain a conditions for the new relations, with the purpose that under expansions by these relations requisite features are preserved.

These features can be hereinafter: model completeness (for example expansion of weakly o-minimal model complete theories), expanded model elementary theory decidability (for instance, exponential function expansion of the theory of the real numbers field, open question), strong minimality, omega-stability, superstability, stability, o-minimality, weak o-minimality, finite cover property, no independence property, and others.

Researches in second half of the preceding century occasioned in the complete theories segregation into classes, subject to the nature of definable sets and their systems.

In the preceding decades expansion challenges are suitable for all classes of complete theories. Systematization of complete theories is displayed in Picture 1:

Picture 1[[1]](#footnote-1) - Complete Theories

Major experts in model theory got extensive results in distinct challenges of expansions (expansion by elementary substructures, automorphisms, nonelementary substructures, non-indiscernible sets etc.).

Strongly minimal theories (B.I. Zilber, E. Hrushovski, A. Baduisch, B. Poizat, E.A. Palutin, J. T. Baldwin, K. Holland, V.V. Verbovskiy, S. Buechler, A.T. Nurtazin, A. Pillay, M. Macintyre, B. Baizhanov - J. Baldwin etc.).

-stable (A. Nesin, A. Borowik, B. Poizat, G. Cherlin, B. Zilber, J. Baldwin - K. Holland, A. Baduisch and others).

Superstable (E. Bouscaren, T.G. Mustafin, B. Poizat, B Baizhanov - B. Baldwin - S Shelah, E. A. Palyutin, A.A. Stepanova and others).

Stable (B. Poizat, E. Bouscaren, J. Baldwin - M. Benedikt, Kazanova - Ziegler, B. Baizhanov - J. Baldwin, K. Kudaibergenov and others).

O-minimal (L. Van den Dries, A. Wilkie, Ch. Steinhorn, A. Pillay, D. Marker, D. Macpherson, E.A. Palyutin, S. Starchenko, Peterzil, B. Baizhanov, E. Baisalov - B. Poizat and others).

Weakly o-minimal (D. Macpherson – D. Marker – Ch. Steinhorn, E.A. Palyutin, B.S. Baizhanov, V.V. Verbovskiy, B.Sh. Kulpeshov, R.D. Arephyev, R. Vencel and others).

Quasi o-minimal (O. Belegradek - A. Strobushkin - M. Taitslin, J. Baldwin - M. Benedikt and others).

Quite o-minimal (B. Kulpeshov, V.Versbovskiy).

Dependent (NIP) theories (S. Shelah, H.D. Macpherson – D. Marker – Ch. Steinhorn, J. Baldwin, M. Benedikt, A. Pillay, F. Wagner, V. Verbovskiy and others).

Simple (S. Shelah, E. Hrushovski, E.A. Palyutin, B. Poizat, N. Kim, A. Pillay, M. Macintyre, F. Wagner, V. Kolesnikov, Vasilyev - Itay, Pillay - Poltakovskaya and others).

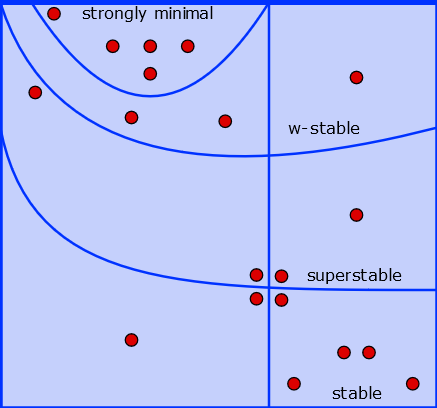
-stable (E.A. Palyutin, T.G. Mustafin, B. Poizat, T. Nurmagambetov, A. Stepanova and others).

Numerous outcomes were revealed in the upcoming classes of complete theories: dependent theories, weakly o-minimal theories, o-minimal theories, stable theories, superstable theories, o-stable theories, strongly minimal theories.

The upcoming approach the expansion problem can determined:

Let be individual classes of complete theories, if is a model of some complete theory , then which considerations on new relations are necessitous and/or sufficient for the purpose of having be a model of theory ?

Consider results in alpha quadrant in Picture 2.



Picture 2 – Alpha Quadrant

Strongly minimal theories.

A theory is titled  *strongly minimal* if in every single model every definable set is finite or negation of this definable set is finite (1971) [9]. A problem of unary function expansions of an algebraically closed field, to such an extent that this function is an automorphism was examined by A. Macintyre. B. Zilber began to investigate the systematization of strongly minimal theories as chunk of a investigation of the uncertainty of finite axiomatizability and the spectrum of complete theories. Zilber suggested a hypothesis on the geometry of strongly minimal theories. Hypothesis: For strongly minimal theories the geometry arising from the algebraic closure operation is either one of the upcoming types: expansions of an algebraically closed field, trivial or locally modular. E. Hrushovski (1988) built an illustration of a strongly minimal non-locally-modular theory, so that this theory can not be interpreted in a field, and in the theory a group is not interpreted. That is a counterexample to Zilber’s hypothesis. V.V. Verbovskiy verified for that example that its elementary theory doesn’t accept exclusion of imaginaries [10, 11] (2002, 2006). In 2004 B.S. Baizhanov and J.T. Baldwin showed: for all strongly minimal theory the upcoming holds true: expansion by an arbitrary set is stable (superstable) whenever the strongly minimal formula has trivial geometry [12]. However in 2004 J.T. Baldwin and K. Holland established possibility of unary predicate expansion of an algebraically closed field to such an extent that structure obtained after expansion would be omega-stable of Morley rank for any natural [13].

Omega-stable theories.

Bruno Poizat in his research articals published in The Journal of Symbolic Logic in 1999 and 2001[14, 15] constructed finite Morley rank omega-stable field, which has two arbitrary unary predicates. John T. Baldwin and Kitty Holland found out that it has non-model-complete teory. In their 2004 work published in journal “Annals of Pure and Applied Logic” [13, P. 159] J.T. Baldwin and K. Holland revealed sufficient requestes, that a strongly minimal model has a model complete theory in case by unary predicates for every -categorical expansion. In example constraction Hrushovski put notions “pre-dimension and dimension” of finite structures as foundation. When he built omega-stable theory, of a finite extension of the designated finite structure, Hrushovski determined dimension as pre-dimension. In 2003 premised on the conception of topological space completion, V.V. Verbovskiy has developed a pre-dimension defining technique on one class of infinite structures, which has enabled to transfer to the study of generic stable structures on the manner of studying generic omega-stable structures [16].

Superstable theories.

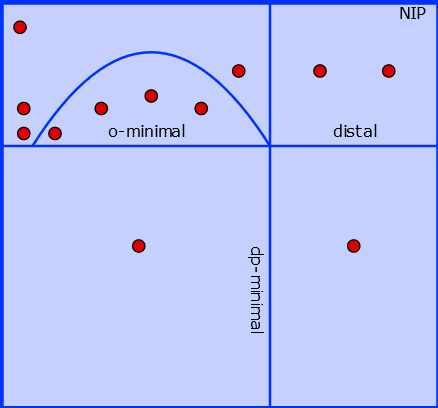
For superstable theories Elisabeth Bouscaren examined sufficient expansions of models by unary predicate establishing an elementary substructure.

In 1989 she verified that an absense of the dimension order property in the initial superstable theory implies stability (superstability) of a pair of models is stable (superstable) and vice versa [17]. In 1988 E. Bouscaren and B. Poizat verified that in stable theories this outcome does not possess [18], by building an example of non-superstable theory, where pairs of models theory is complete and stable and dimension order property possess. In 1989 part of Bouscaren’s [17, P. 205] evidence was that she verified that in the expanded language the types identity over the two tuples follows from the types of the tuples in small model identity in the original language. Suchlike characteristic is considered being *benign*. In 2005 J. Baldwin, B.S. Baizhanov and S. Shelah have verified that sameness of strong types of the original language tuples over random set of superstable model causes sameness of types over expanded language tuples [19]. Such characteristic is considered being *weakly benign*. It is thus confirmed weakly benignity of each and every superstable theory sets.

Stable theories.

Couples of stable theory model, to an extent that large model is satiated over a submodel were examined by Bruno Poizat. He named such a couple of models lovely pair. In 1983 he verified that there is no finite cover property in the initial theory whenever the theory of lovely pairs is complete [20]. In 2000 steadiness of a satiated expansion of model of stable theory by predicate, which discern a non-definable set were proved by Baldwin and M. Benedikt [21]. Results from [21, P. 4937] were enhanced by E. Casanovas and M. Ziegler in 2001, by getting rid of the indiscernibility, and establishing non finite cover property in terms of this set [22]. The outcomes of Baldwin-Benedikt-Casanovas-Ziegler and the outcomes B. Poizat and E. Bouscaren of were enhanced by B.S. Baizhanov and J. Baldwin, by verifying that the weakly-benign set constraint of an expanded model has stable theory whenever the weakly benign expansion of model preserves stability (resemblant outcome takes place for the class of superstable and -stable theories as well). Moreover in 2004 Casanovas-Ziegler’s question on designation of the characteristics of the model having finite cover property in regards to set was answered by B.S. Baizhanov and J.T. Baldwin [12, P. 1243].

Consider the beta quadrant in Picture 3.



Picture 3 – Beta Quadrant

O-minimal theories.

The dynamic investigation of linearly ordered theories founded on the concept of o-minimality has started since the middle of 80’s. In 1996 one of the significant results is real numbers field unary expansion, which is o-minimal, decidable and admits quantifier elimination, by exponential function has a model completeness property and it’s theory is o-minimal was verified by Alex Wilkie [23]. In 1997 van den Dries, A. Macintyre, D.Marker showd that every single o-minimal structure on produces a family of definable sets that has local triviality, stratification and property of uniform finiteness and is quite stable under various topological / geometrical operations [24]. The property that every single structure elementary equivalent to a structure linearly ordered is o-minimal whenever a structure linearly ordered is o-minimal as well verified by Ch. Steinhorn, A. Pillay and J. Knight. Furthermore in 1986 and 1988 they clarified definable functions characteristics [25-27]. In 2007 K.Zh. Kudaibergenov enhanced the Marker’s outcomes in o-minimal theories about small extensions of models in [28]. In 2007 B.S. Baizhanov verified that there exists an elementary extension to extent that the unary partial functions class with definable parameters is not identical to the unary partial functions class, defined over original language parameters whenever ordered-minimal expansion of a theory model that is dense ordered-minimal and accepts QE is essential [29].

Let , to such an extent that for every single definable set of structure . When investigating new formulas of the expanded model for the formula of the initial language, the formula of the expanded language broaches two questions linked to the -type of the model of the original theory:

Will the set be consistent?

Is the -type over the set A be realized in the model?

In some cases, positive answers to these questions suggest the nature of formulas of the expanded model: lovely pairs of models (B. Poisat [20, P. 239]), small indiscernible set (Baldwin-Benedikt [21, P. 4937]), small set without the finite cover property with respect to this set (Casanovas-Ziegler, [22, P. 1127]), lovely pairs (Pillay-Vassiliev [30]), H-structures (Berenstein-Vassiliev [31]).

The notion of geometric theories were originated by E. Hrushovski and A. Pillay in [32] (1994). It is a regular comprehensiveness of the classes of dense o-minimal and strongly minimal theories in as much as it admits the elimination of “there are infinitely many” quantifier property and the algebraic closure exchange property, which is geometric theory. Since 2010 A. Berenstein and E. Vassiliev have been studying unary predicate expansions of geometric theories, such that predicate has extension and density properties as well as investigating interconnections of the properties of the initial theory and the properties of expanded theories [31, P. 866; 33-34]. Their work is basen on the concept of simple theory models – lovely pair. Belle pair – lovely pair of models of a simple theory was examined by Bruno Poizad. In [30, P. 491; 35-38] A. Berenstein and E. Vassiliev developed and studied it.

Weakly o-minimal theories.

The French-Kazakh Colloquium on model theory in Almaty in June 1994 was accompanied by extensive amounts of prominent scientists. An american scientist Charles Steinhorn pontificated on the Colloquium with a report on o-minimality. Since then the collaboration of B.S. Baizhanov and other Kazakh colleagues with Ch. Steinhorn arose.

Charles Steinhorn has sent article draft of D. Macpherson, D. Marker, Ch. Steinhorn [39] on weak o-minimality and a parcel of copies of works on o-minimality by J. Knight, A. Pillay, C. Steinhorn [25, P. 565; 26, P. 593; 27, P. 469], L. Mayer, D. Marker, Ch. Steinhorn, A. Pillay, [40-43] Kazakh scientists resolved all the problems stated in the works on weak o-minimality [39, P. 5435].

External definability.

D. Macpherson, D. Marker, Ch. Steinhorn approach. In 1994 it was verified by D. Macpherson, D. Marker, Ch. Steinhorn (in the draft [39, P. 5435]) that for an o-minimal structure expansion by predicate preserves weak o-minimality in case that such predicate is “unary convex predicate” and it is crossed by a uniquely realizable cut. According to D. Marker [40, P. 63] 1-type uniquely realizable over a model M, that is if for every single realizing this -type , is the only realization of in , has the corresponding characteristic: there is no definable function operating on the type’s set of realizations. Two structures simultaneously was considered by Macpherson-Marker-Steinhorn: and , that is a satiated elementary extension and a model of o-minimal theory. A new unary convex predicate was defined with an element that is the realization of type , an irrational type, to such an extent that the following holds for every *d from M:*

.

The signature formulas are constructed by induction. It implies a signature formula exists to an extent that for any *from M* the following holds:

.

The occasion was the decisive point in this construction. They defined

.

Although -type is uniquely realizable, 1-formulas over convex to the right and the left from have solutions outside . Hence if , so for certain for every single ,

.

It means the previous formula segment holds on , namely

Hence there exists to such an extent that

Consequently .

So for every single --1-formula the set of all its realizations in will be a convex sets finite union, , owing to the fact that consists of finite union of points and intervals. Elementary theory of ’s is weakly o-minimal being that the number of convex sets is not infinite and because of this doesn’t depend on parameters.

There was made a systematization of the non-orthogonality of -types theory in [25, P. 565; 40, P. 63; 41, P. 146; 42, P. 185]. On this basis in 1995 it was found out that the case of is type that is non-uniquely realizable. (Pillay-Steinhorn, D. Marker, L. Mayer, Marker-Steinhorn, 1986–1994), B.S. Baizhanov proffered [44] to take the constants for from an indiscernible infinite sequence over where is from . When there exists a finite number of irrational cuts in other words one-types over M, to such an extent that for every single such one-type , the subset of

there exists an -1-formula , such that

There was obtained the one-types systematization over “weakly o-minimal theory” model subset by B.S. Baizhanov in 2001 [8, P. 1382]. He solved a problem of expanding a “weakly o-minimal theory” model by an unary convex predicate. The monotonicity characteristic for definable functions on weakly ordered-minimal structures was verified by R.D. Aref’ev in 1997 [45]. Sample of a weakly ordered-minimal structure, whose theory is not weakly ordered-minimal was built by Macpherson-Marker-Steinhorn [39, P. 5435]. Also, in 2001, “a weakly o-minimal” ordered group example, which theory isn’t “weakly o-minimal” was built by V.V. Verbovskiy [46]. In 1998 B.Sh. Kulpeshov developed a designation of a linearly ordered structure weak ordered-minimality in terms of the set of realizations 1-types convexity in the examining of “weakly ordered-minimal structures” [47], and he made a complete characterization of linear orders that is weakly o-minimal, he established the concept of unary formula convexity rank , that is useful in investigating countably categorical structures. Also in 2007 and 2011 he found a touchstone for binarity of weakly o-minimal countably categorical structures in expressions of types binarity and convexity rank [48, 49]; In 2006 he proved the binarity and characterized the structures of convexity rank 1 that are countably categorical weakly o-minimal [50, 51]; Also in 2011 and 2013 he characterized quite o-minimal countably categorical structures [52, 53]. There was introduced a designation of behavior of -preserving to the left convex and to the right convex 2-formulas in “weakly o-minimal theories” by Bektur S. Baizhanov and Beibit Sh. Kulpeshov in 2006 [54]. Starting from 2006 B.Sh. Kulpeshov in the series of works has given the criterion for binarity of “weakly o-minimal” countably categorical theories in the point of view of “convexity rank” and binarity of every non-algebraic 1-type, a complete characterization of countably categorical “weakly ordered-minimal finite rank of convexity” theories, and an entire countably categorical “quite o-minimal theories” characterization [49, P. 354; 51, P. 185; 52, P. 387; 55-56] (2006-2016).

Since 2018 successive results in a number of work on the field of expansions of “weakly o-minimal structures” were published by B.Sh. Kulpeshov and S.S. Baizhanov [7, P. 673 ;57-58]. In [57, P. 106] B.Sh. Kulpeshov and S.S. Baizhanov proved that convexity rank and -categoricity of an expansion of a “weakly o-minimal” -categorical theory of finite “convexity rank” is preservated in case of finite set of unary convex predicates expansion. In [58, P. 207] authors have developed a criterion for preserving both weak o-minimality and -categoricity in case by an relation of equivalence expanding of 1-indiscernible weakly ordered-minimal -categorical structures. In [58, P. 207] they also developed a criterion for preservation of the -categoricity of a weakly o-minimal 1-indiscernible expansion by a binary predicate on countably categorical weakly o-minimal 1-indiscernible structures of rank-1 convexity. In [7, P. 673] it was developed a touchstone the countable categoricity of a weakly ordered-minimal expansion of 1 rank of convexity under expansion by every single binary predicate of weakly o-minimal non 1-indiscernible countably categorical structures is preserved.

The concepts of a weakly quasi-o-minimal model and theory were established and inspected by K.Zh. Kudaibergenov [59] (2010). In 2012 and 2013 K.Zh. Kudaibergenov established and inspected numerous o-minimality extensions into “partial orders” [60, 61]. It was proceeded generalizations of the o-minimality concept by various ways by K.Zh. Kudaibergenov. In 2018 K.Zh. Kudaibergenov established and inspected the notions of right o-minimality, multi-R-minimality, and their variants [62].

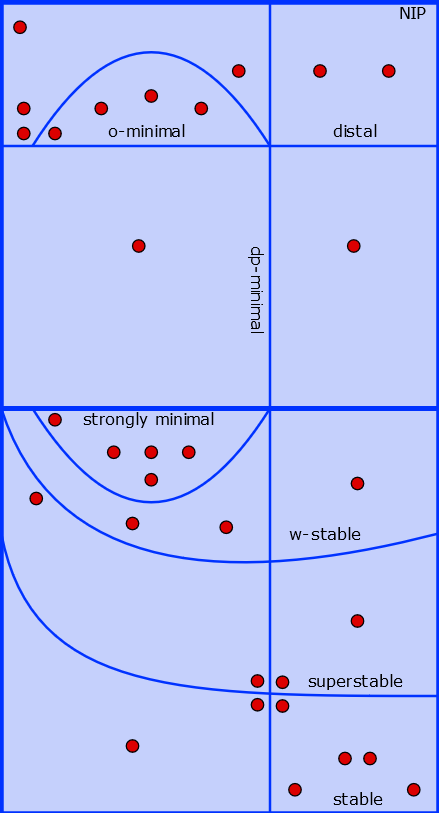
“Countably categorical weakly ordered-minimal theories” were investigated by H.D. Macpherson, B. Herwig, A.T. Nurtazin, G. Martin and J.K. Truss and they verified that every 3-indiscernible model is -indiscernible for every single natural . They built samples to such an extent that there is a 2-indiscernible model, which is not 3-indiscernible [63] (1999).

In a series of papers on the field of circularly ordered structures B.Sh. Kulpeshov (2006-2016) [64-67] obtained a number of results. B.Sh. Kulpeshov fetched complete designation of the behavior of functions definable unaryly for a countably categorical 1-transitive structure weakly circularly minimal [64, P. 555]. He characterized up to binarity countably categorical “weakly circularly minimal structures” with a non-primitive 1-transitive automorphism group [65, P. 282]. B.Sh. Kulpeshov developed a touchstone that 1-type of convexity rank 1 realizations is indiscernible in non-1-transitive “weakly circularly minimal” countably categorical structures [66, P. 255]; He defined almost binarity of countably categorical non-1-transitive “weakly circularly minimal theories” [67, P. 38].

There was established characterization up to binarity of “weakly circularly minimal structures” countably categorical with a primitive automorphism group by B.Sh. Kulpeshov and H.D. Macpherson [68] in 2005. High homogeneity of every 6-homogeneous weakly circularly minimal countably categorical structure was verified by them. In 2015 an any weakly circularly minimal cyclically ordered group is abelian was verified by V.V. Verbovskiy and B.Sh. Kulpeshov [69]. There was acquired a complete description of “weakly circularly minimal” non-1-transitive “countably categorical -convex” (where ) almost binary of convexity rank 1 theories by A.B. Altayeva and B.Sh. Kulpeshov [70] in 2016.

*Type is named definable if for every formula there is a controlling formula that ⬄ holds.* Definability of every single type over every set is the key characteristic of a stable theory was acquired in 1978 by Shelah [71], that is, for any type and for every formula there is a controlling formula if and only if an elementary theory is stable. Extension is said to be *-conservative* if the type of any -tuple of elements of over is definable where is an elementary extension of a model . There was verified by L. van den Dries [72] in 1984 that any type over the field of real numbers is definable. And it is true that every elementary extension is conservative. In 1994 this result was applied and enhanced to the class of “o-minimal theories” by David Marker and Charles I.Steinhorn. They verified that from 1-conservativity of a couple of models of an “o-minimal theory” it folows verifying their -conservativity [42, P. 185]. At a late date Anand Pillay re-proved this result. He realized that the “o-minimal theory” is axiomatizable for conservative pairs of models [43, P. 1400]. This result relates to the class of “weakly o-minimal theories”, however in 2005 it was disseminated to a broader class by B.S. Baizhanov. He constructed non-2-conservative pair of models of “a weakly minimal theory” that is 1-conservative [73, 74]. In 2007 B.S. Baizhanov established the conservative pairs models of “weakly o-minimal theories” axiomatizability condition and verified that for any model of “a weakly o-minimal theory”, excluding an discrete linear order theory o-minimal expansion by maximal and minimal elements, there exists a saturated conservative elementary extension [75].

Consider a wider class of complete theories, which is union of alpha and beta quadrant in Picture 4.



Picture 4 – NIP Theories

Dependent theories (NIP theories).

A problem of systematization of dependent theories appeared since the end of 90s. Proper subclasses (“weakly o-minimal”, “o-minimal”, and “quasi-o-minimal” classes of complete theories) of the dependent theories class were obtained [42, P. 185; 43, P. 1400]. B.S. Baizhanov and V.V. Verbovskiy, Based on the methodology developed in stable theory, ascertained the class of ordered stable theory. In 2011 verified that the o-stable theory class is a dependent theory’s subclass and that the pure linear order theory is o-superstable [76]. Definability of one-types for o-stable theories was explored in [77] in 2015 by B. Baizhanov and V. Verbovskiy. O-stable ordered groups and fields were investigated by V.V. Verbovskiy. He verified o-stable ordered group commutativity. Additionally, he characterized definable subsets, and constructed a vast amounts of non-trivial o-stable ordered groups examples [78] (2012). S. Shelah represented the notion of a dp-minimal theory in the frame of systematization of dependent theories . There was verified of o-stablity for “dp-minimal theories” with a definable linear order by V.V. Verbovskiy [79] in 2010. There was proposed the notion of “a theory stable up to “ by Viktor V.Verbovskiy [80] in 2013. He verified that NIP theories are “stable up to a certain formulas subset” without the independence property [80, P. 119].

In 2013 using stability up to it a characterization of NIP theories was shown and the concept of relative stability was introduced by V.V. Verbovskiy [80, P. 119]. He proceeded to investigate definability of types and relatively stable theories in [81] and verified that for a stable up to delta theory it holds that its delta part is definable if and only if every single one-type over a model of is definable. V. Verbovskiy proceeded to study ordered o-stable groups in 2018. He built a sample of an ordered group with “Morley o-degree” at most 4 of Morley o-rank 1 and showed that every Morley o-rank 1 ordered group with definable convex subgroups, that is boundedly many, is “weakly o-minimal” [82]. In 2015 and 2018 circularly ordered groups were investigated in the articles [69, P. 82] and [83] respectively. The Abelian property of weakly circularly minimal groups was first checked by Beibit Kulpeshov and Viktor Verbovskii. Next it was amplified up to the circularly ordered class of o-stable groups by V. Verbovskiy.

The first-order theories independence property was investigated by K.Zh. Kudaibergenov. In 2011 he disproved the existence of infinite indiscernible sequences of big cardinalities models of NIP theories, that is the Shelah’s hypothesis strong form [84]. In 2013 he refuted the Adler’s claim. Kanat Zh. Kudaibergenov built a NIP theory, in which atomic formulas without independence property [85].

**2 EXPANSIONS OF MODELS BY UNARY PREDICATES**

**Definition 2.1** If every definable subset is a finite “union of convex sets” then totally ordered structure is called *“weakly o-minimal”* [8, P. 1382].

**Definition 2.2** [8, P. 1382] If is “a weakly o-minimal structure”, , is -satiated, are non-algebraic 1-types then it is said that type is “*non weakly orthogonal” to* () if there exists a formula and a realisation that there exists realisations and the following holds: and .

**Lemma 2.1** [8, P. 1391] *The relation of “non-weakly orthogonality” is an equivalence relation on for B subset of K, where K is a “weakly o-minimal structure” .*

*Proof:* Reflexivity is obvious, for any type it is non-weakly orthogonal to itself: if we consider we can use formula , in that case there are such that and , where is any other realisation of and . Let be two 1-types such that . Then is not complete type. Thus such that is not complete type. Then . Let be 1-types such that and then as if we consider formula and such that and and and where without loss of generality can be either or . Let be equal to . Then if we conisder a formula we get and which is

In [39, P. 5435] some concepts originally were introduced. We recall them. Suppose following statements hold: is an –definable set; is a projection in which the last coordinate throw out; ; for every takes place

Let the set bounded above, but has no supremum in for every . We denote as a relation of –definable equivalence on , and define it by the next way:

Suppose that , and as we denote the -class of the tuple for every tuple . It exists a natural order–definable linear on , which determined next way. Let and . Then for any . If , then there is some that or and thus induces a linear order on . The set we call a *sort* in (a –definable sort in in this exact case), where is a Dedekind structure completion. And is considered as naturally embedded in . The same way, it possible to get a sort in , taking into consederation infinum instead of supremum.

**Definition 2.3** [39, P. 5435] Function  is called locally decreasing (locally constant or locally increasing) on where *B* is infinite subset of *N*, *L* is subset of and *N* is structure linearly ordered if for each single there exists an infinite interval containing on which is strictly decreasing (constant or strictly increasing).

If *f* is locally decreasing or locally increasing on *B*  then *f* is called locally monotone on a set .

Take an -definable function on and an -definable equivalence relation on . A function is strictly increasing on whenever for all such that and the following holds .

A function is called to be strictly decreasing on whenever for all such that and the next takes place .

**Definition 2.4** [47, P. 1511] is designation of the *convexity rank* of for -definable formula with one free variable where *N* is a sufficiently saturated model of , is “a weakly o-minimal theory” . C*onvexity rank* of is defined by following way:

1. For infinite :

2. If an equivalence relation parametrically definable and an infinite family so that

- for all and ,

- and is a convex subset of for any

then

3. If for all with a limit ordinal

then

is called *determinable* if such that Otherwise, i.e. for any .

Denote by the *convexity rank* of a 1-type . is the minimal convexity rank of formulas from type .

**Proposition 2.1** [52, P. 387] *If domain of function includes into an -definable sort where is a “weakly o-minimal structure”, , then the function is either locally constant or locally monotone on .*

**Theorem 2.1** [56, P. 606] *Let be a model of categorical theory which is a weakly o-minimal with finite convexity rank, with . The following holds:*

(i) There exists a finite set (or when doesn’t have a last or a first element) which consists of all -definable elements of except possibly and such that for any and for any either there is no element from between and or there is a dense linear order without endpoints, i.e. or is a dense linear order which doesn’t have endpoints, moreover for some and .

(ii) Every non-algebraic type has some convexity rank , where is integer, . This means there exists an empty definable equivalence relations: , , , such that

- is partitioned by into infinitely many open and convex -classes. On these classes the induced order is a dense linear order which doesn’t have endpoints.

- for every every -class is partitioned by into infinitely many open and convex -classes. The -subclasses of every -class are linearly ordered dense which doesn’t have endpoints.

(iii) For all nonorthogonal, nonalgebraic types with the following holds:

(1) if the definable closure of some realization of contains some realization of , that is for some then there exists a unique -definable function which is: locally monotone bijection on whenever , locally constant on whenever that is on each -class is constant and on f is locally monotone.

(2) if the definable closure of any realization of doesn’t contain any realization of , that is for arbitrary then whenever there exists exactly -splitting formulas , , such that for every . The function is locally monotone on

Whenever there exist exactly -splitting formulas , , such that for every . The function is locally monotone on and constant on each

hence, admits quantifier elimination to the language

where for each , the formula isolates the type . Furthermore there is a weakly o-minimal -categorical theory which has finite convexity rank corresponding, as above for every ordering with elements that can be distincted as in (i)-(iii).

**Definition 2.5**  For a weakly o-minimal structure M, to extent that . Assume that that types are non-algebraic and is -saturated. In case of existance of bijection , where *f*  is a function over A, we call type is not  *quite orthogonal* to type (). In the case of coincidence of concepts of weak and quite orthogonality of 1-types we are going to say that an weakly o-minimal theory is *quite o-minimal*.

As soons as in the case of o-minimal theory for every single set A and every single two types over A there is a bijection that is A-definable and strictly monotone between sets of realisations of such types it is clear that any such theory it is also quite o-minimal [86].

**Example 2.1** [39, P. 5441] Consider a structure that has a property of linear order and the universe consits of unary predicates and interpretations more precisely their disjoint union, with . We identify the interpretation with the set of rational numbers , that is ordered as usual, and interpretation is , that is ordered lexicographically. The symbol *f* defines a partial unary function to extent that and interpreted, and the following equality holds for all .

It can be verified that has a weakly o-minimal theory. Consider the following types , . It is clear that types and with , so we can conclude that is not quite o-minimal. Also note that .

**Example 2.2**  Consider a structure that is linearly ordered and to extent that is the ditinct union of and unary predicates realisations, to extent that realisations ordered as follows . Thus realisations of predicates and is similar to the , ordered lexicographically. Two binary predicates and are interpreted as equivalence relations on and respectively. Relations of quvalence for every , are defined by next way:

For

A symbol is interpreted by a partial unary function with and . It is determined as follows: for every .

It can be understood that and are partitioned into an infinite number of infinite convex classes by -definable equivalence relations and respectively. We are saying that function is strictly decreasing on any , where , and on function is strictly increasing. It is easy to prove, that is “a quite o-minimal theory”. A convex set defined by set is not an interval in . For this reason the theory is not o-minimal. Note also that .

Quite o-minimal theories are a subclass of weakly o-minimal theories which has many o-minimal theories properties. In [53, P. 45] it were described examples of quite o-minimal countably categorical theories. Their binarity follows from this description (o-minimal countably categorical theories has a similar result).

**Theorem 2.2**  *Suppose is a countably categorical theory quite o-minimal, and where Thereat the following holds* [52, P. 390 ; 53, P. 48]:

(i) there is a finite set , whensoever hasn’t first or last element), which consists of all empty elements definable in (with eventual exceptions for , such that for any and for all either or is “a dense linear order without endpoints” and there exists and that ;

(ii) let is any non algebraic one-types, then there exists such that , so that there is empty definable equivalence relations , , , that the following holds

- an induced order on the classes is “a dense linear order without endpoints” because partitions into an infinite number of convex and open classes

- for any splits up every -class into an infinite number of convex and open classes , so that the set of subclasses of every class is “densely linearly ordered without endpoints”

(iii) a relation of equivalence exists, such that any non-algebraic 1-types over empty set arbitrary enumeration is , and for each there is a unique locally monotone empty definable bijection that , and for all to such an extent that accepts quantifier exception up to the language such that insulates the type for each .

Moreover, any ordering of chosen elements as mentioned in (i)-(ii) and any appropriate relation of equivalence as mentioned in (iii) meets to a quite o-minimal countably categorical theory as set out above.

**Definition 2.6** [87] Suppose is a weakly o-minimal structure, , is -satiated, is non-algebraic.

(1) It is called that an -formula is *-preserving* (or *-stable*) if there exists the type , , realisations such that

and .

(2) It is called that a formula is *convex to the right (left)* if there is -stable and there exists that is convex and the is in and it is the left (right) endpoint of .

**Definition 2.7** [54, P. 31] A formula is called to be is *equivalence-generating* if there is -preserving convex to the left (right) and for each realisations such that , we have the following:

**Lemma 2.2** [54, P. 33] *Suppose is a weakly o-minimal structure, , is -satiated, non-algebraic type , is a -preserving convex to the right (left) formula. If , is not equivalence-generating there exists realisations , such that*

**Lemma 2.3** [54, P. 34] *Suppose is “a weakly o-minimal structure”, , is -saturated, non-algebraic type , convex to the right (left)-preserving formula . In this case a formula is also convex to the right (left) -preserving.*

**Lemma 2.4** [54, P. 36] *Suppose is a “weakly o-minimal structure”, , is -saturated, is a equivalence-generating formula-preserving convex to the right (left). Then*:

1) is a formula -preserving convex to the left (right) and is also equivalence-generating.

2) is a relation of equivalence that splits up into infinitely many infinite convex classes.

**Proposition 2.2** [54, P. 37]*Suppose is a countably categorical “weakly o-minimal theory”, N is a model of theory T, , non-algebraic one-type . Then any convex formula -preserving to the right (left) is equivalence-generating.*

### 2.1 Unary expansions

Suppose is a model of “weakly o-minimal theory”. For some formula denote the set as . Denote the set .

Suppose is a model of “weakly o-minimal theory”, , . Denote the set as . Denote the set .

For any weakly o-minimal countably categorical structure any expansion by a new convex unary predicate preserves weak o-minimality, that is has “weakly o-minimal theory” [8, P. 1382]. It is worth noting that an expansion using an unary predicate with interpretation which is a finite quantity of convex sets in , say , is equivalent to the expansion by a finite number of unary convex predicates , for , because all these convex sets are -definable. As is countably categorical there exists only a finite number of non-algebraic 1-types over . Call them . Without loss of generality assume:

Let lie across , , so that there exists and such that

Then the introduction of the unary predicate is equivalent to the introduction of two convex unary predicates and . Hence we will consider a unary predicate such that and for some non-algebraic one-type , it means that there is that .

If the right boundary of is determined by some , then this expansion is definable, non essential, and equivalent to extension of by one constant. Obviously all the properties of initial model is preserved in this case and keeps being countably categorical. Therefore further we consider the case, that the right boundary of doesn’t lie in and hence determine irrational cut in .

Suppose an is a relation of equivalence definable over empty set, that splits out into infinite number of classes eash of them is infinite and convex. A predicate is irrational with respect to the -classes whenever the following holds:

(1) for each for which takes place, there exists a realisation where

(2) for each for which takes place, there exists a realisation where

Let’s see the next sample.

**Example 2.3**  Consider a linearly ordered structure , where is a unary convex predicate in . Replace each element by a copy of the set of rationals and define new binary relation by next way: for any they are in the similar equivalence class if their first coordinates match, that is

We obtain the structure , which is splited out into infinite number of convex classes by the equivalence relation , so that the order induced on the -classes is a dense linear order which doesn’t include endpoints.

is a weakly o-minimal countably categorical structure. The predicate is irrational with respect to the -classes.

is called quasirational to the right (left) with respect to the -classes there exists and there is an -class that

**Lemma 2.5** *Let type be non-algebraic, and relation of equivalence definable over empty set splits out into infinite number of classes which are convex and infinite. In the case when with respect to the -classes is quasirational to the right (left) there is an empty definable quasirational to the left (right) convex formula with respect to the -classes.*

*Proof:* It is possible to expect that is quasirational to right point with respect to the -classes, as proof for the case of quasirational to the left is similar. Hence there exists with . Consider the formula

So is quasirational to the right (left) with respect to the -classes empty definable convex formula.

**Theorem 2.3** [58, P. 207] *Suppose for some is a model of a countably categorical “weakly o-minimal theory” of convexity -rank, and is the expansion of by a finite family of convex unary predicates , where . Then the theory also is a countably categorical “weakly o-minimal theory” of convexity rank* .

In order to prove Theorem 2.3 we will require several lemmas. We assume for now that is a weakly o-minimal countably categorical theory which has convexity of finite rank, and .

**Lemma 2.6**  *Given non-algebraic type with , the predicate partitions a number of type realizations into -definable 1-indiscernible convex sets, where .*

*Proof:* Since is a countably categorical theory, the type is isolated, hence there exists an -definable isolating formula . As , there is relations of equivalence definable by parametrs and the relations split out into infinite number of classes which are infinite and convex such that holds

for some .

Relation of equivalence splits out into ordered by type infinite number classes which are infinite and convex. Every class is splited out into infinite number of -subclasses which are convex, ordered by type , . Each -class is 2-indiscernible over . Therefore it is sufficient to study the mutual location of -classes and the predicate , where .

Consider the formulas:

Case 1. Predicate defines an irrational cut with respect to the -classes. For this case doesn’t have realzations in for all and is divided into two convex formulas: and .

Case 2a. For some each class is divided by the predicate and is irrational with respect to the -classes. In this case is divided into the formulas:

where .

Case 2b. Some -class is divided by . In this case is divided into the formulas:

Case 3. For some each -class is divided by the predicate and is quasirational to the right with respect to the -classes. In this case divided into the formulas:

where .

Case 4. For some each class is divided by the predicate and is quasirational to the left with respect to the -classes. In this case is divided into the formulas:

where .

**Lemma 2.7** *Let be two non-algebraic one-types over the empty set, such that and . If is partitioned into -definable convex sets, then is also partitioned into -definable convex sets.*

*Proof:* Let be an isolating formula of . As , there exists a -splitting formula such that the function is locally monotone on [56, P. 606]. We list, in each of the following cases, the convex formulas partitioning .

Case 1. is irrational with respect to the -classes. Lemma 2.6 shows that is divided into the two formulas

and

If increases strictly on then

In case if is strictly decreasing on then

Case 2a. For some some -class is divided by predicate and is irrational with respect to the -classes.

In case when f is strictly increasing on

If is strictly decreasing on

Case 2b. Some -class is divided by predicate . If is strictly increasing on then

If is strictly decreasing on then

Furthermore, if is strictly increasing on for some , where , then

In the case when is strictly decreasing on for some , where

Case 3. For some some -class is divided by the predicate and is quasirational to the right with respect to the -classes.

Case 4. For some some -class is divided by the predicate and is quasirational to the right with respect to the -classes.

**Lemma 2.8** *Let be two non-algebraic one-types over the empty set, such that and . If is partitioned into -definable convex sets, then is partitioned into -definable convex sets, where .*

*Proof:* Let and . There exists an -splitting formula such that is constant on each -class [56, P. 606], furthermore is the greatest equivalence on with this property and is locally monotone on where

Case 1. is irrational with respect to the -classes. The existence of a -splitting formula implies that is divided into two formulas and . An is strictly monotone on . Thus if strictly increasing on then

If is strictly decreasing on then

Case 2a. Some -class is divided by the predicate for some and is irrational with respect to the -classes. In this event is partitioned by into -definable convex sets. In the case when the formulas and , for , are defined as in Lemma 2.6 case 2a. If , where , then additionally we define the formula

It’s clear that for some . Formula is taken off if for every . Therefore number of formulas are at most .

When the formulas , for , are defined as in Lemma 2.6 case 2a, and there are at most of them.

Case 2b. Some -class is divided by . In this case and we obtain formulas: and , where , if for each . Otherwise the formula is added.

Case 3. Some -class is divided by for some and is quasirational to the right with respect to the -classes. In the case when we obtain formulas: and , where . In the case when we obtain formulas: where and for m from interval: .

**Lemma 2.9**  *Suppose are two one-types non-algebraic which are over the empty set, such that and If is partitioned into -definable convex sets, then is partitioned into -definable convex sets, where .*

*Proof:* There exists a -splitting formula such that the function is constant for all -classes [56, P. 606], furthermore is the greatest equivalence on with this property and is locally monotone on where

Case 1. is irrational with respect to the -classes. Existence of a -splitting formula implies that is divided into two formulas and . An is strictly monotone on . Thus if strictly increasing on then

If is strictly decreasing on then

Case 2a. -class is divided by the predicate for some and is irrational with respect to the -classes. In this event is partitioned by into -definable convex sets. In the case when on - is strictly increasing holds

In the case when on - is strictly decreasing holds

In the case when is strictly increasing on for some , where

In the case when is strictly decreasing on for some , where

Hence we obtain number of formulas.

Case 2b. Some -class is divided by . In this case we obtain formulas: and , for .

Case 3. Some -class is divided by for some and with respect to the -classes is quasirational to the right. formulas are obtained in this case: where and , where .

Now we can prove Theorem 2.3.

*Proof:* Let . Based on Lemma 2.6 a number of implementation of is partitioned by into s -definable 1-indiscernible convex sets, where . Using Lemmas 2.7-2.9 a number of implementation of every non-algebraic 1-type , with , is also partitioned into -definable convex sets, where , and each of these sets is a number of implementation of one-types over in the expanded model. Define their convexity ranks:

Case 1. is irrational with respect to the -classes. Let

It is clear that and with

From this point and forward we omit the indication of weakly orthogonal pairs among these types

Case 2a. -class is divided by where and is irrational with respect to the -classes.

Suppose

and for .

If

(1) then and Then , and

where

(2) If . Let . If , then

for .

If such that then the formula additionally appears and so is the type . It is clear that . If for every then this formula doesn’t appear. Also

for ,

and

for .

In case of the situation is identical to case (1).

(3) If . Let . In this case

for . Note that

where .

Case 2b. divides -class. Suppose and where .

(1) . In this case

and

(2) If , then

for . If exists such that then the formula additionally appears and so is the type . It is clear that . In case of for each the formula doesn’t appear. Note that

for , and

for ,

(3) In case of Let , then

for .

Note that

for .

Case 3. -class is divided by for some and is quasirational to the right with respect to the -classes.

An example for the types where *m* is from interval . Suppose and . Thus and

So we shown that in any expansion by a unary convex predicate there always exists at least one non-algebraic 1-type such that , moreover this convexity rank is maxiaml.Thus we conclude that and theory have the same convexity ranks. Also note that the -splitting formula (or a -splitting formula in case ) for non-weakly orthogonal types is a -splitting formula for new non-algebraic non-weakly orthogonal types . And finally from the Theorem 2.2 it follows that is countably categorical.

Further, we use the obtained result to study the properties that are preserved during the expansions of models of a theory which is quite o-minimal and countably categorical by a predicate convex unary. Next is determined: properties that are preserved during such expansions are quite o-minimality, countable categoricity, and convexity rank.

**Lemma 2.10**  *Let be quite o-minimal countably categorical theory, , non-algebraic one-types , such that . Suppose that is an expansion of a model by a unary predicate such that and . Then is partitioned into convex -definable sets in is partitioned into convex -definable sets in .*

Proof of Lemma 2.10. As then by Theorem 2.2(iii) and there exists -definable function , which is a locally monotone bijection. Let be an -definable formula, isolating type . Suppose that is divided into convex -definable formulas , selecting in sets which are 1-indiscernible over . Consider the following formulas:

It is evident that for each distinct pair with , as , we have . Due to indiscernibility over each will also be 1-indiscernible..

Let for any . Then it is clear, that , and is -definable bijection.

Thus taking in consideration Theorem 2.3 we establish the following:

**Corollary 2.1** *Suppose is a model of theory which is a quite o-minimal and countably categorical, and be an expansion of a model by an arbitrary finite family of predicates convex and unary. Then is a model of theory which is a quite o-minimal and countably categorical of the same convexity rank.*

**3 EXPANSIONS OF MODELS BY EQUIVALENCE RELATIONS**

The section will examine expansions of countably categorical weakly o-minimal theories by special case of binary expansions – expansions by equivalence relation.

is called an  *1-indiscernible* structure if for every .

**Example 3.1** [63, P. 65] Suppose , where is the set of -tuples of rational numbers, ordered lexicographically by , and suppose for each the equivalence relation be given by for any . Then for every the equivalence classes of are convex subsets of . Moreover, refines for every .

It can be shown is an 1-indiscernible countably categorical weakly o-minimal structure and *Theory of Nm* has *RC(Nm)=*.

**Proposition 3.1** [49, P. 354] *Suppose is an 1-indiscernible countably categorical weakly o-minimal structure of finite convexity rank. Then there is that is isomorphic to (Example 6.1).*

In this case we examine only the problem of preserving both weak o-minimality and countable categoricity for expansions of models of 1-indiscernible countably categorical weakly o-minimal theories of finite convexity rank by a relation of equivalence spliting the universe of the model into infinite number of infinite convex classes.

**Example 3.2**  Suppose is a linearly ordered structure on the rational numbers set . It is evident is a countably categorical o-minimal structure. We expand the model by a new binary relation next way:

suppose is such that for every and exists :

Then it is easy to understand is a relation of equivalence that splits into infinite number of infinite convex classes, and the -classes are ordered by the type .

It is a routine to show using simple quantifier elimination that is a weakly o-minimal structure. It should note that is not countably categorical because the ordered set of integers is interpretable as .

**Example 3.3**  Suppose is a linearly ordered structure on the set , ordered lexicographically. The relation is defined next way:

It is evident is a relation of equivalence that splits into infinite number of infinite convex classes, and the -classes are ordered by the type .

We extend the universe of the structure by adding two elements to every -class, which are the left and the right endpoints of the -class. As a result, we get a new structure . Consider the reduct of the structure to the structure . It is evident is a countably categorical o-minimal structure. Its expansion is a countably categorical linearly ordered structure.

We consider the next formula:

The formula means that is some -class endpoint. It should note that is an union of infinite number of convex sets. Thus, is not weakly o-minimal.

**Proposition 3.2**  *Suppose is an 1-indiscernible countably categorical weakly o-minimal structure of convexity rank 1, is an expansion of the model by an equivalence relation spliting into infinite number of infinite convex classes. For to be a countably categorical weakly o-minimal theory necessary and sufficient condisions are the next statements to hold:*

(1) It exists only finite number of -classes having at least one endpoint;

(2) It exists only finite number of -classes having an immediate predecessor or an immediate successor in the induced ordering on .

Proof of the Proposition 3.2. (Necessary condition). We consider the formula from Example 3.3. It is clear that is finite. If was infinite then it would contain an infinite interval because of weak o-minimality, but endpoints of infinite convex -classes can’t form an infinite interval. Therefore, is finite, which means that it exists only finite number of -classes having at least one endpoint.

Now we prove that condition (2) holds. Assume the contrary: it exists infinite number of -classes having an immediate predecessor or an immediate successor.

*Case 1*. ( it exists a discretely ordered chain of -classes of length .

Then it exists a model of the by compactness, in which there is an infinite discretely ordered chain of -classes. We don’t lose generality if suppose that such chain is ordered by the type . We consider the next formulas:

It should note that defines the class , and for every defines the class and the -classes immediately following it.

Then we obtain that there exists such that

which contradicts countable categoricity of .

*Case 2*. with a discretely ordered chain of -classes of length and is maximal with this property.

At that point (j such that and there is an infinitely many chains of length . Thereat such that is a union of infinite number of convex sets, which controverts weak o-minimality of .

(Sufficient condition) According to (1) it exists only finite number of -classes which have at least one endpoint. Now, each the endpoint is definable in obedience to the linear ordering of . According to (2) we can also define using a formula every -class which has an immediate successor or an immediate predecessor, as well as possible nonempty intervals between some of these classes (those intervals in which -classes are densely ordered without endpoints); In addition, minimal -classes (the leftmost -class) or maximal -classes (the rightmost -class) in intervals with dense ordering of -classes are distinguished. In consequence, we obtain finitely many -definable formulas , so that for all

All the formulas define some 1-type over . Using standard methods it is not difficult to understand that up to atomic formulas and the formulas admits quantifier elimination (the last formulas define convex sets in ), therefore we get that is a countably categorical weakly o-minimal theory.

**Corollary 3.1** *Suppose is an 1-indiscernible countably categorical weakly o-minimal structure of convexity rank 1, is an expansion of the model using an relation of equivalence which split into infinite number of infinite convex classes. Now for to be an 1-indiscernible countably categorical weakly o-minimal structure necessary and sufficient condisions are :*

(1) All -classes have no endpoints in ;

(2) The induced order on -classes is a dense linear order without endpoints.

**Example 3.4**  Suppose is the structure from Example 3.2. We replace all points using copy of rational numbers and define a new structure , in which the relation defines by next way:

Now it is easy to show that is a relation of equivalence which splits each -class into infinite number of infinite convex classes that the -subclasses of each -class are densely ordered without endpoints.

It can be proved that is a weakly o-minimal structure, but the is not countably categorical.

**Example 3.5**  Suppose is ordered by type countable number of copies of the structure from Example 3.2. Now we get a new structure , in which the relation is defined by next way:

At that point for all and is a relation of equivalence which splits into infinite number of infinite convex classes, ordered by type . Observe that -subclasses of each -class are ordered by type .

It is also easy to understand that is a weakly o-minimal structure, but is not countably categorical.

**Theorem 3.1**  *Suppose is an 1-indiscernible countably categorical weakly o-minimal structure of convexity rank , and , are -definable relations of equivalence which split into infinite number of infinite convex classes, that for all*

Suppose is a model expansion using a new relation of equivalence splitting into infinite number of infinite convex classes. Now for to be a countably categorical weakly o-minimal theory *necessary and sufficient condisions are :*

(A) Only finite number of -classes having at least one endpoint exist;

(B) Only finite number of -classes having an immediate predecessor or an immediate successor in the induced ordering on exist;

(C) is splitted into finite number of infinite convex sets such that for each exactly one of the next items holds:

that for all ;

, and

that ,

for all ;

, , and

that for some and

that for some , and

such that for some ,

and .

Proof of Theorem 3.1. (A) and (B) ensue from the Proposition 3.2 proof. Further, we consider the following formulas for any and :

i.e. defines the set of elements that .

i.e. defines the set of elements such that , and .

i.e. defines the set of elements such that , , , , *N* and .

i.e. defines the set of elements such that , , and .

i.e. defines classes for some element such that

i.e. defines classes for some element such that

i.e. defines an -classes for an element that

i.e. defines an -classes for an element that

i.e. defines non-empty intersections for some that there exist and where .

i.e. defines non-empty intersections for some such that there exist and with .

We can understand that for all there exist such that , , and , and also that

for all single that , , provided that , or .

In accodance with weak o-minimality of , each of these formulas defines a set that is the union of finitely many convex sets, which implies (C).

In obedience to the proof of Proposition 6.2 performance of (A) and (B) conditions define using a formula the available endpoints of -classes; any -class, which has an immediate successor or an immediate predecessor, as well as gaps where -classes are densely ordered without endpoints; in addition, minimal or maximal -classes are distinguished in the intervals of dense ordering of -classes having the leftmost or rightmost -class. Therefore, we obtain finitely many -definable formulas , that for each

The performance of condition (C) provides that any formulas , , , , defines a set that is the union of finitely many convex sets. Then by linear ordering of all formulas decomposes into finitely many convex -definable formulas for some .

It is clear that there exist only finite nuber of -classes (and, therefore, -classes) defined by next formulas:

i.e. defines classes for some which are immediately followed by an -class.

i.e. defines classes for some which are immediately followed by an -class.

Suppose opposite: there exist infinitely many -classes and -classes defined by the formula . Then we sate that is the union of infinitely many disjoint convex sets. Actually, if , then there exists such that

and . As far as the condition (B) holds, we have that it exists an infinite convex part of satisfying the formula , whence is the union of infinitely many disjoint convex sets, but it contradicts the condition (C).

At the end, it easy to proved using standard methods that accepts quantifier elimination up to atomic formulas and formulas

, , , ,

from which we have is a countably categorical weakly o-minimal theory.

**Corollary 3.2** *Suppose is an 1-indiscernible countably categorical weakly o-minimal structure of convexity rank , and are -definable equivalence relations splitting into an infinite number of infinite convex classes, such that for all c from N*

Suppose *N* is model expansion by a new relation of equivalence which splits into infinite number of infinite convex classes. Thereat for *N* to be an 1-indiscernible countably categorical weakly o-minimal structure *necessary and sufficient condisions are*:

every -class has not endpoints in ;

the induced order on -classes is a dense linear order without endpoints;

for all *c from N* exactly one of the following items holds:

that ;

, and

that ,

, and

## 4 EXPANSIONS OF MODELS BY ARBITRARY BINARY PREDICATES

The section is devoted to investigation of the question of properties preservation when expanding models of countably categorical weakly ordered-minimal theories by arbitrary binary predicates. Earlier we have studied the problem of preserving properties for expansions of models of countably categorical weakly o-minimal theories by unary predicates. As it is known, in work [8, P. 1382] B.S. Baizhanov proved that the expansion of a model of a weakly o-minimal theory by a unary predicate that distinguishes a finite number of convex sets preserves weak o-minimality of the expanded theory. However, in the case of expanding a model of a weakly o-minimal theory by a binary predicate that distinguishes a finite number of convex sets for each fixed both the first and the second parameter, the expanded theory can lose weak o-minimality (Example 4.1).)

**Example 4.1** Suppose is a linearly ordered structure on the set of real numbers . It is evident that is a model of a countably categorical o-minimal theory. We expand the model by a new binary relation next way: suppose be such that is the graph of the next unary function , defined as for every *d from*  and for every . It is evident that and for every are singleton sets, i.e. convex sets. However, notice that is not weakly o-minimal, so far as there is no partition of the set into a finitely many convex sets, on each of which the definable function is locally constant or locally monotone.

**Example 4.2** Suppose is a linearly ordered structure on the set of rational numbers . It is evident that is countably categorical 1-indiscernible o-minimal structure. Consider a binary predicate ) expansion of a structure : denote as that for all

We understand that and are convex for all . It easy to prove that is weakly o-minimal 1-indiscernible structure.

The formula is convex to the right -stable, where

It is obvious that the formula doesn’t generate equivalence.

Study next formulas:

For every we got

It means that is not countably categorical.

Suppose is a weakly o-minimal structure, , a non-algebraic type , an -definable formula , that is -preserving, that is for all *c from* there exist elements that the next holds .

Owing to the weak o-minimality of the set composes of a union of convex sets, whose quantity is finite. It is evident that every single of sets mentioned is -definable. To the left of element there is a finite quantity of such definable convex sets. Signify them as , in the following way

In the same way to the right of element there is a finite quantity of such definable convex sets. Signify them as , , , in the following way

In the event that definable convex set has an element . Denote the set as . Thereby if , then so that .

Determine next formulas:

It is evident that the formulas , are convex to the right -stable and formulas are convex to the left -stable.

A formula is said to be *equivalence generated*, if all nontrivial formulas from the set

, , , , ,

are equivalence generating formulas.

**Example 4.3** Suppose is a linearly ordered structure on the set , ordered lexicographically. It is evident that is countably categorical o-minimal structure.

Represent the next binary formulas and on the set : for all

Suppose

and is an expansion of model using binary predicate . It is evident that for all is convex and .

It easy to understand that is 1-indiscernible weakly o-minimal structure, nevertheless is not countably categorical theory.

Examine the next formulas:

The formulas are -stable convex to the right, where

is equivalence-generating, and is not equivalence-generating. Therefore the predicate is not equivalence-generating predicate.

**Theorem 4.1** *Suppose N is 1-indiscernible countably categorical weakly o-minimal structure of convexity rank 1, is 1-indiscernible weakly o-minimal expansion of a structure using binary predicate .*

Thereat for to be countably categorical *necessary and sufficient condisions is* thefulfillment next properties:

(1) and is equivalence generated;

(2) For each empty definable equivalence relation , generated by a predicate , the set of -classes is ordered densely.

Proof of Theorem 4.1. *(Necessary condision).* Let be countably categorical theory. Examine a predicate . Through weak o-minimality of the structure for every and are a union finite quantity of convex sets. According to Proposition 2.2 both formulas and must be equivalence-generated.

Suppose is a random empty definable equivalence relation. Through 1-indiscernibility the set of -classes must be either densely ordered without endpoints, or discretely ordered without endpoints. Hence, by countable categoricity, the set of -classes must be densely ordered.

(S*ufficient condisions).* Suppose and are formulas generated equivalence. Examine a random -definable equivalence relation, generated by a predicate . In accordance with condition, the set of -classes is densely ordered. Through 1-indiscernibility there is neither the leftmost -class, neither rightmost -class. Also through 1-indiscernibility there doesn’t exist -class, having at least one endpoint (if every -class would have at least one endpoint, then we would get contradiction to weak o-minimality of ).

Through the weak o-minimality of a structure for all elements the sets and are unions of finite quantity of convex sets. Hence, it exists only a finite quantity of formulas of the form , , , , , , , for some . As in accordance with the condition , are equivalence generated formulas, then every non-trivial formula from next:

generates an equivalence relation. Thereby we get only finite quantity of -definable equivalence relations, generated by predicate .

Suppose are a complete set of -definable equivalence relations, generated by predicate . Through 1-indiscernibility there don’t exist such that , and for some ,

Also there don’t exist such that for some

Onward for some if there is such that , then for all . Thereby it exists (possible situation when for some ) and perhaps some renumbering of the existing equivalence relations so that for all we would have

As far as, in accordance with condition, the set of -classes is densely ordered for each -definable equivalence relation , then -subclasses of every -class are densely ordered without endpoints, where and

Onward we can establish using standard methods, that accepts quantifier elimination up to atomic formulas and formulas , , where do we get that is countably categorical.

Example 4.4 shows, that there exists an expansion by binary predicate, that preserves weakly o-minimality but doesn’t preserve countable categoricity.

**Example 4.4**  Consider a structure that is linearly ordered, the universe is defined as following , where and are matched with and have the same order as , where is a set of rational numbers. To distinguish the elements of these sets, for any element denote its identical element in as .

An unary function with and is defined by the symbol as following , so strictly increasing bijection is defined from to by . It can be shown that is a weakly o-minimal ω-categorical structure.

Consider model expansion by new binary relation . Consider , where for any

It clear, that is still weakly o-minimal structure. Examine next formulas:

and for any

so *Theory M'* isn’t countably categorical theory.

**Example 4.5**  Consider a linearly ordered structure such that the universe , with and matched with and had the same order as , where is a set of rational numbers. To distinguish the elements of these sets, for any element settle it’s identical element in as . Symbol is designate by next way: for any

It can be shown, that is weakly o-minimal ω-categorical structure.

Also examine the model expansion using binary relation: suppose , in which for all

It is clear, that remains weakly o-minimal structure. Examine next formula:

Defining formulas analogically to previous example we obtain that is non countably categorical.

The following example shows that under the expansion of a weakly o-minimal ω-categorical structure by -splitting formula for non-algebraic types we can lose as 1-indiscernibility of this types and weakly o-minimality of such expansion.

**Example 4.6** [47, P. 1511] Consider a linearly ordered structure , where is a union of interpretations and that are disjoint and , with identified with the , ordered lexicographically, and identified with and had the same order as , where is a set of rational numbers. We define using simbol *f* a partial unary function for which and by next way: , for every

It is obvious that is a weakly o-minimal ω-categorical theory, and

determine a relation of equivalence, splitting into infinitely many convex classes.

Let . Then it’s clear that are non-algebraic, .

Consider the following formulas:

It is clear that formulas and are -splitting formulas and for any .

Consider an expansion of a structure by binary predicate such that for every and the formula below is true:

Then the set is convex for every ,

and .

Consider the following formula:

It is evident, that the set is the union of infinite number of -separable convex sets, so is not 1-indiscernible and and non weakly o-minimal.

There was given a complete description of countably categorical theories which have finite convexity rank in the work [56, P. 606]. Since in weakly o-minimal theory of 1 rank of convexity there is no equivalence relation with infinite number of convex classes, then as a corollary we get:

**Corollary 4.1**  *Consider a weakly o-minimal countably categorical theory T of convexity rank 1, . The following is true:*

(i) Exists an infinite set (, if is not having the last or the first elements), made up of every single empty definable element from (except if there is some of them) that for all and for each either or is linear order, that is dense, without endpoints and there exists and such that ;

(ii) For any nonalgebraic types such that

If for some , then there is the unique empty definable function , such that it is strictly monotone bijection on

If for all , then there exists the only -splitting formula , such that - strictly monotone on

Thus a theory accepts exception of quantifiers to the language

Where isolates type for every

Moreover for any ordering with selected elements, as represented in (i)-(iii), related to weakly o-minimal ω-categorical theory of convexity rank 1, as it was presented earlier.

Suppose is weakly o-minimal, countably categorical of convexity rank 1, and are nonalgebraic 1-types in over emptyset. Suppose is an expansion of a structure by binary predicate , such that for any the set is convex, and . Suppose also that is weakly o-minimal structure and .

The following theorem gives necessary and sufficient conditions of theory countable categoricity.

**Theorem 4.2** Consider a model *of a weakly o-minimal ω-categorical theory of 1 rank of convexity, non-algebraic one-types and are from . Let be an weakly o-minimal expansion of convexity rank 1 of a structure by binary predicate , such that , for any the set is convex, and . Thereatis countably categorical .*

*Proof:* () Suppose is ω-categorical. By countable categoricity there exists an empty definable formulas and , such that and .

Note that as is weakly o-minimal, countably categorical theory of convexity rank 1, by corollary 7.1 for any either these types are weakly orthogonal or there exists the only -definable bijection and there is no other relations between them or there exits the only one -splitting formula and there is no other relations.

Towards a contradiction suppose that types are nonweakly orthogonal in structure . Hence there exists -splitting formula , so for any the set is convex, and .

As theory has convexity rank 1, then the function , is strictly monotone on . Without loosing the generality suppose that is strictly increasing on .

Let . As has convexity rank 1 also, , then is also strictly monotone on .

Case 1. is strictly increasing on .

Without loosing the generality suppose, that for some the following holds . Then for any also holds.

Consider the following formulas:

Then for any the following holds:

so is not countably categorical. It contradicts suggestion.

Case 2. Suppose strictly decreases on .

Thereat that and . The last contradicts that

Let . Let’s show is an ω-categorical.

Case 1. For some the set has right endpoint.

So consider the formula below:

It is clear that . By the assumptions , from where we get that . Thus for any the set has the right endpoint.

Hence the function is strictly monotone bijection between and .

Case 2. For some the set does not have right endpoint.

So consider the formula below:

It is clear that . By the assumptions , from where we get that . Thus for any the set does not have the right endpoint.

hence the formula is the only -splitting formula.

In both cases admits quantifier elimination to language

Where is set of all -definable elements of structure . isolates nonalgebraic 1-type for every ; is the only -definable bijection between and , which is strictly monotone. is the only -splitting formula. Then by the corollary 7.1 theory is countably categorical.

The following example shows that an expansion of ω-categorical weakly ordered minimal structure of convexity rank greater than 1 by -splitting formula for nonalgebraic types can preserve countable categoricity even in the case when this types are non weakly orthogonal.

**Example 4.7**  Revising example 4.6 introduce the following changes: The interpretation of is ordered lexicographically, and let partial unary function is defined by equality for every .

It is evident, that is similarly weakly ordered minimal ω-categorical structure.

By the assumptions for each there is such that where .

Consider an expansion of structure by binary predicate such that for any and the following holds:

Then we also have that the set is convex for every

and .

It is possible to see is weakly ordered minimal ω-categorical theory and .

Examine the formula below:

It is possible to see, that is convex to the right p-stable formula, moreover formula is equivalence-generating. Thus by Lemma 2.4 the formula

is a relation of equivalence which splits into infinitely many infinite classes and for all type realisation the extending is faithful . Wherefrom we have that convexity rank of type equals to 3 and this implies that the theory has RC=3.

**5** **EXTERNAL DEFINABILITY AND MODEL COMPLETENESS**

Let , be two structures of the signature , such that . We say that a set is  *externally definable*, if for some -formula, . In stable theory any externally definable set is internally definable. If we take some family of externally definable sets such that it closed over all set theory operations: union, intersection, taking complement, cartesian product, projection, we obtain family of externally definable sets. Expansion by such family of externally definable sets is called to be  *pure externally definable expansion*. Notice that main moment in the expansion by externally definable sets to be pure externally definable expansion, is that this family closed over the operation of projection [39, P. 5435] (preprint 1994).

The idea to consider the expansion by externally definable set belongs to Dugald Macpherson, David Marker and Charles Steinhorn and expansion of model by externally definable set was introduced for the first time in 1994 in preprint of [39, P. 5435]

### 5.1 Model completeness

**Definition 5.1.1** *Suppose is a model of weakly ordered minimal theory , are its large satiated elementary extension of , .*

We say that an one-type is solitary if any -2-formula , such that for any , if then . In case when is a model of o-minimal theory, , defines irrational cut, this one-type is called to be uniquely realizable [40, P. 63].

We say that an one-type is quasi-solitary if there is a 2-formula over A that for every , and for every 2-formula over M, that for every , whenever the . In the event when we have solitary one-type.

Notice that it follows from the definition that is convex.

The following theorem was proved for an o-minimal case in [39, P. 5435].

**Theorem 5.1.1** [39, P. 5435] *Let be a model of a weakly ordered minimal theory such that any type over is solitary. Then any expansion of by a unary convex predicate is pure externally definable expansion and has weakly o-minimal theory.*

*Proof:* Consider - a model of a weakly ordered minimal theory of signature , be the expanded signature of , and be the theory of .

Convex expansion always go through in general two cuts. Expansion of model by convex predicate can be considered step by step as expansion through each cut. We don’t lose generality if we consider such expansion, that next sentence holds:

Let be an elementary extension of such that there exists a realization of one-type over M which defines cut of expansion. So, we suppose that .

Claim: For any of signature formula of the signature so that for every the proceeding is true:

Proof of Claim: It will be examined the claim by induction method on depth of formulas constructions.

Consider the depth 0 formula. For it

where are the signature formulas. For the formula we can define

Then for any the following holds:

Suppose that relation holds for the formulas of depth n. Consider the following formula

Further proof is identical to a given in historical review for o-minimal theories by Macpherson-Marker-Steinhorn.

**Theorem 5.1.2** [88] *(Model completeness of expansion by unary predicate, solitary)*

Let be a model of weakly o-minimal model complete theory. Expansion of model by unary predicate , where , is theory of .

Let , such that , since is model complete, .

If goes through solitary type then . Thus is model complete.

*Proof:* Let be convex to the right formula for any such that

Claim: Theory

Proof of Claim: We will prove using method by contradiction. Suppose that the statement for is not true, then

As far as for some we have

Let , . Then and . Consider set of -1-formulas . By , is consistent and has unique extension to complete solitary one-type over , which determines cut and the expansion. So any realization of satisfies solitary one-type one side and on other side by . Contradiction.

Claim says that for any model of , the cut determined on is solitary. Since , . The last means that . Thus we can take the realization of the cut for as in proof of Theorem 8.1 and convenable for the proof for . For any of theory and that there is of theory and such that

The same holds for :

is the same formula for both and . Whenever we consider

thus is model complete.

**Theorem 5.1.3** [89]*(Model completeness of expansion by unary predicate, quasisolitary) Let be a model of a weakly o-minimal, model complete theory, be an expansion of model by unary predicate , such that goes through quasi-solitary type, where . Then*

(i) is weakly o-minimal and this expansion of is pure externally definable expansion.

(ii) is model complete, if greatest -preserving 2-formula -definable

(iii) model complete theory, where is tuple from the greatest -preserving 2-formula.

*Proof:* This repeats proof of the same theorem for case solitary. The proof consists in showing by induction on the construction of a formula in the expanded language that the hold on the parameters is equivalent to the hold of the corresponding formula

in the initial language on these parameters and an additional parameter from the saturated elementary extension. For any of signature it exists formula with , of the signature so that for all the next holds:

There are small change in the formulas (1) and (2). Let be the greatest -preserving formula. If in solitary case i.e. by formula , then in case quasi-solitary by formula .

Let be convex to right formula of signature satisfied , then the property " is -preserving" is expressed by -formula:

be -formula that says that is more than any -preserving:

Let the set of all convex to right -formulas with condition . Then define two sets of -sentences. and . Since for any convex to right -formula holds: and , whereat , whereat the parameters is the ones of the most great -preserving 2-formula. So far as , if we consider arbitrary model of as an expansion of then we obtain that this expansion goes through quasi-solitary 1-type with greatest -preserving , . Then for any of the signature formula with , of the signature so that for all the proceeding is true:

Consider such that . Since is model complete . If is -definable -formula, then for any of signature , for any , because the greatest -preserving in both models coincide

Thus is model complete.

Consider such that . Since is model complete , . If is -definable -formula, then for any of signature , for any , because the greatest -preserving in both models coincide

Thus is model complete.

### 5.2 External definability

Externally definable sets is an special case of expansion, which is equivalent to extension, i.e. any formula in new language is defined in initial language using new external parameters.

*External definability.* Let be elementary substructure of . The pair of models , such that is saturated over , is called  *beautiful* pair. Let , where . Then we define the predicate on the set , where is arbitrary formula, if the next holds:

Let denote by . If a pair of models is conservative pair (that is type of any tuple elements from over is definable), then from the definition the structure is the structure obtained from scolemization of . We will consider two approaches of the simple cases when constructed from one -type for o-minimal theory.

Consider the signature complete theory . Denote a model of the theory as . The expansion using type is , if , where .

If for every single formula of the signature there exists a formula of the signature and such that the proceeding is true:

We say it this occasion that  *admits uniformly representation of* -formulas by -formulas,

In 2000 [39, P. 5435] it was verified that unary convex predicate expansion of o-minimal structure preserves weak o-minimality, in the case when this predicate is traversed by a 1-type, that has a unique realisation, by Macpherson-Marker-Steinhorn. Uniquely realizable -type over M, as introduced by D. Marker [40, P. 63], is the only the -type realization from the set of the type realization over a prime model M and this realisation of type. The following characteristic is true for uniquely realizable -type *p*: for the number of realizations there is not any definable functions that can act on it. At the same time Macpherson-Marker-Steinhorn approach was descirbed in historical review, and was applied for uniquelly realisable, solitary, quasi-solitary, model complete theories.

*B.S. Baizhanov approach.* For the event when is a non uniquely realizable type, on the base of theory of (non)orthogonality of -types and its systematization made in [25, P. 565; 40, P. 63; 41, P. 146; 42, P. 185] (Marker, Mayer, Pillay-Steinhorn, Marker-Steinhorn, 1986–1994), B.S. Baizhanov proposed [44; P. 3] (1995) to take the constants from an infinite indiscernible sequence over for , where from . Taking attention that there is a finite quantity of irrational cuts (that is -types over ) ifthe set , that for every such -type , is a subset of

there exists an -1-formula , such that .

There is two parts of the idea to use an indiscernible sequence.

(i) If , consequently for every tuple , , ), , when and have identic type over .

(ii) But to find a sequence such that for each ,

for every (), .

To find the sequence that is indiscernible define the properties ()-() that follow from the theory of non orthogonality of -types over sets in o-minimal theories and the systematization of -types.

() [40, P. 63] (Marker 1986). Consider two one types , and let type be non complete 2-type (In 1978 Shelah verified that such and are non weakly orthogonal). In this case there exists an monotonic bijection definable over and therefore, is irrational whenever is irrational;

Moreover is uniquely realizable whenever is.

Recall that whenever type is irrational is convex non-definable set without maximal and minimal elements.

() Whenever is irrational for any , , where

,  *there exists an* -1-formula , , .

The  *quasi-neighborhood of* in () is the union of sets definable over , and every this definable set is a subset of , a convex non-definable set without minimal and maximal elements. Therefore such definable set is a subset of ( *neighborhood of*  in ). This explains the equality of two convex sets.

() If is non uniquely realizable and irrational, then for any , whenever the 1-types and are non uniquely realizable and irrational.

By theorem of compactness and () there exists such that and since is non uniquely realizable that is there exists definable over monotonic bijection , neighbourhood can not have minimal and maximal element. Concerning that for irrational type , is a convex non-definable set without maximal and minimal elements, and are irrational and because , where acts on and . This means and are non uniquely realizable.

Denote . Then by (), () is non uniquely realizable, irrational and finitely satisfiable in because the right sides of and coincide

For any ,

(1)

and therefore, all these sets have empty intersection.

Proof of (1) is done by induction on . Assume (1) for m.

Let and

Suppose that

As long as the first set does not change, there exists a formula definable over such that

Let be the endpoint of one of the intervals of formula , then since is non uniquely realizable,

As long as and by () there exists monotonic function definable over such that . However and therefore, there is 1-formula over such that

Then belongs to set definable over . This means . Contradiction.

From () and (1) it follows that for every , whenever for any , and

(2)

Let for -formula corresponding formula of signature is , , . Therefore to have the solution in for any formula it is sufficient to write the formula

In 1996 B.S. Baizhanov dealed with problem on model of weakly ordered minimal theory concerning the unary convex predicate expantion of a model. He submitted in the Journal of Symbolic Logic his results of obtaining a systematization of 1-types over a subset of a model of weakly ordered minimal theory. His arlicle was published in 2001 [8, P. 1382]. Ye.Baisalov and B. Poizat studied "beautiful" pairs of ordered minimal theories models. In 1996 they verified the quantifier elimination [90]. Hard to say that are the approaches in [90, P. 570] and [44, P. 3] are different, because they have similar principles (i)-(ii) from [44, P. 3].

We say that the expansion by all externally definable subsets admits quantifier elimination, whenever for every formula of signature there exists a formula of signature , and the element that for every the proceeding is true:

*Approach of Shelah.* In 2004 [91] a model of NIP theory was examined and quantifier elimination holds true for the expansion by all externally definable subsets that means that it is NIP was verified by S. Shelah. The biggest issue is the "there exists in the submodel" quantifier elimination. Towards a contradiction he proposed an indiscernible sequence and shown that "there exists in the submodel" quantifier elimination for failure implies that is true whenever for some is even from that it can be proven that the theory has IP.

In 2005 Shelah’s simplified proof was found by V.V. Verbovskiy and F. Wagner [92], in short by using the notion of a finitely realizable type. In 2006 Shelah’s theorem additional two re-proofs were given by A. Pillay using the notions of quantifier-free heirs and types and for the other re-proof he used the notions of quantifier-free coheirs and types.

The analysis of approaches shows that we can control the number of one-types realizations using the theory of orthogonality [93]. The generalization of notions of neighborhood and quasi-neighborhood it is possible to formulate the following theorem

**Theorem 5.2.1** Consider a complete theory *that for any set the next holds:*

1) type , for each tuple , that

2) type the proceeding is true. If , then .

Then the expansion by all externally definable subsets of model of the theory admits the elimination of quantifiers.

**Theorem 5.2.2** [94]*Let be a weakly o-minimal ordered group of signature , be an elementary extension of , i.e. . Suppose that such that is irrational, and is an expansion of by , such that it admits a uniform representation of -formulas by -formulas. Then preserves both weak o-minimality and group properties.*

*Proof:* Consider an expansion of a weakly o-minimal group by predicate , such that

Let’s show that this expansion is externally definable.

Consider an arbitrary quantifier free formula of signature . This formula can be represented in disjunctive normal form:

where and are atoms in language . Then we can swap every with , thus we get quantifier free formula such that

Continue by induction. Let

holds for any formula of depth . Consider a formula of depth . We should find a formula , such that whenever . By theorem 58 from [8, P. 1382] , such that for any formula there exists a formula such that

Consider a formula , the set is a definable in weakly o-minimal group structure. Thus it is union of finite number of convex sets. does not increase (in case some convex sets are cover irrational cut, two convex sets stick into one convex set). Thus every single definable set in is union of finite number of convex sets, and it follows by the definition that it is weakly o-minimal.

**CONCLUSION**

The dissertation considers expansions of models of NIP theories. Such theories include the following: linearly ordered theories, countably categorical theories, weakly o-minimal theories, theories of finite convexity rank. The aim was to investigate preservation of certain properties(countable categoricity, weak o-minimality, convexity rank) of models by expanding by unary predicates, equivalence relations or binary predicates. The key results in this context are as follows:

Expansion of model of countably categorical weakly o-minimal theory of finite convexity rank by a finite family of convex unary predicates preserves countable categoricity and convexity rank. Similar result for quite o-minimal countably categorical theories: Expansion of model of countably categorical quite o-minimal theory of finite convexity rank by a finite family of convex unary predicates preserves countable categoricity and convexity rank.

More complicated result for expansion by equivalence relations: Touchstone for maintaining both countable categoricity and weak o-minimality (and in addition to this the 1-indiscernibility) when expanding a model of a 1-indiscernible countably categorical weakly o-minimal theory of finite convexity rank by an equivalence relation partitioning the universe into infinitely many infinite convex classes.

Results on expansions by binary predicates: Touchstone for maintaining countable categoricity for a 1-indiscernible weakly o-minimal expansion of countably categorical weakly o-minimal theory of convexity rank 1 by every single binary predicate.

Touchstone for maintaining countable categoricity for a weakly o-minimal expansion of a non-1-indiscernible countably categorical weakly o-minimal theory of convexity rank 1 by random binary predicate.

Maintaining weak o-minimality when expanding a weakly o-minimal ordered group by an externally definable binary predicate.

**Assessment of the completeness of the aims of the work**. The results of investigation are new received using on our own tools and methods. Conditions of preservation of either weak o-minimality, countable categoricity under expansion by unary or binary predicates were found. Consequently, the goals of the work have been entirely accomplished.

**Suggestions on applications of the obtained results**. Obtained results in this field of model theory can be used throughout the study of models of NIP theories, particularly expansions of weakly o-minimal theories. Results on the expansions by externally definable sets can be applied to theories of algebraic structures.

**Assessment of scientific level of the work in comparison with the achievements in the scientific direction.** The findings obtained in accordance with the best contributions of foreign colleagues are not lacking and add to the study of the expansion of models of NIP theories..

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1. http://www.forkinganddividing.com/ [↑](#footnote-ref-1)